

Deuteron electromagnetic properties and the viability of effective field theory methods in the two-nucleon system

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Abstract

The central tenet of effective theory is that the details of short-distance physics will not have a significant impact on low-energy observables. Here we perform an analysis of electron-deuteron scattering at low momentum transfers which is based on effective field theory. We show that in our approach the deuteron electromagnetic form factors F_C and F_M indeed are largely insensitive to the short-range NN potential at momenta Q up to about 700 MeV. We also find that the effective field theory approach to deuteron electromagnetic structure provides a systematic justification for many features which have been seen in potential model calculations of the same quantities.

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1 Introduction

The physics of light nuclei would seem to be fertile soil for the application of effective field theory techniques. These techniques rely on the existence of a hierarchy of scales. Once such a hierarchy is established the central tenet of effective field theory (EFT) is that the physics at energies much lower than some scale Λ cannot depend on the details of the dynamics at that scale. The physics at these higher energies appears implicitly in the low-energy effective theory, encoded in a series of contact operators. Suppose then that there are a succession of scales in the problem:

$$\Lambda_1 < \Lambda_2 < \Lambda_3 \dots \quad (1)$$

If we gradually increase the energy being studied we can imagine modifying the effective theory to explicitly include the new physics which emerges as these new scales Λ_n are successively reached.

Light nuclei appear to be an example of a system in which this kind of scale-separation exists. First, it is important to note that in light nuclei the scale of binding is “unnaturally” small. For instance, poles in the two-nucleon T-matrix occur at momenta of 8 MeV (in the 1S_0 channel) and 46 MeV (in the $^3S_1 - ^3D_1$ channel). Both of these are well below the next-lightest scale in the problem, m_π . Thus, key scales in these systems include \sqrt{MB} , the characteristic momentum scale for nuclear binding; m_π , the pion mass; and Λ_χ , the scale of chiral symmetry breaking. In the $A = 2$ system we certainly have $\sqrt{MB} \ll m_\pi \ll \Lambda_\chi$, although the characteristic momentum scale inside larger nuclei may be closer to m_π .

If one confines one’s interest to momenta $p \ll \sqrt{MB}$, then an effective field theory Lagrangian containing contact nucleon-nucleon interactions with increasing numbers of derivatives can be employed. This theory has simple power-counting rules since higher-dimensional operators and loops are suppressed by powers of p/\sqrt{MB} [1, 2, 3]. However, an EFT based on such counting has a very limited domain of validity. Indeed, it is tautological that the bound state at energy B cannot itself be discussed in the context of such an effective theory. To be useful in nuclear physics it is essential that the effective theory’s domain of validity be extended well beyond momenta p of order \sqrt{MB} so that it can describe light nuclei. Somehow the non-perturbative physics associated with the low-energy poles in the NN t-matrix must be incorporated in the EFT.

Once that has been achieved we can work in the regime $\sqrt{MB} \sim p < m_\pi$, and discuss the bound state using a theory which need not explicitly include even the lightest meson, the pion. There has been much recent interest in such pionless effective theories for nuclear physics, as they serve to elucidate many issues associated with the treatment of these low-energy bound states in effective field theory [1]–[12]. They certainly simplify calculations, since the only operators which appear in the effective theory produce separable interactions in the two-nucleon system.

The pionless theory is of somewhat limited use, since in many applications momenta $p \sim m_\pi$ are probed. This “higher-momentum” regime is, of course, still a low-momentum regime from the perspective of hadronic physics. The low-momentum effective theory of hadronic interactions is chiral perturbation theory, in which the pions are introduced as explicit fields. They appear as the pseudo-Goldstone bosons of the spontaneously-broken (approximate) chiral symmetry $SU(2)_L \times SU(2)_R$. A consequence of the spontaneous breaking of this symmetry is that all pion-pion or pion-nucleon interactions involve either derivatives or powers of the pion mass. It follows that a systematic power counting in momenta and m_π may be developed in which higher-order interactions of the pions are suppressed by powers of m_π and/or the typical momentum scale in the problem, p . The techniques of chiral perturbation theory thus provide a systematic expansion which yields predictions for the interactions of pions, nucleons and photons at low momenta. These predictions have been successful in the zero- and one-nucleon sectors, and are essentially model-independent,

as they are based only on the symmetries of QCD and power counting. For a detailed review on this subject see Ref. [13].

We would like to also apply these techniques to many-nucleon systems. However, as noted above, in the two-nucleon case the situation is complicated by the presence of an additional scale which does not appear in the one-nucleon case: the characteristic momentum scale of the bound state, \sqrt{MB} . In the $A = 2$ case this is very different from m_π and thus is usually treated as a separate scale in EFT analyses of this problem. Weinberg's proposal for dealing with the emergence of this new scale was to apply the methods of chiral perturbation theory only to graphs in which the energy flowing through all lines was of order $m_\pi \gg B$ [14, 15, 16]. For these graphs the additional low-energy scale B does not result in any complications, since, by assumption, the energy of all legs is significantly above that scale. Thus chiral perturbation theory can be applied to these "two-particle irreducible" graphs without any modifications. The two-particle irreducible objects calculated in chiral perturbation theory must be sewn together with free Green's functions of the multi-nucleon system in order to generate the actual S-matrix elements of the theory. (For reviews of the application of these ideas to the NN potential and probes of light nuclei see Refs. [17, 18].) It appears then that for the S-matrix elements themselves no systematic expansion exists, despite the demonstrable power-counting for objects that appear in their construction.

Such a point of view is similar to that seen when low-energy effective actions are derived using Wilsonian renormalization group arguments (see Refs. [19, 20] for more explicit investigations of this connection). These techniques are widely applied, and have, for instance, been used to obtain improved actions for lattice gauge theory. In all cases the effective action is obtained using a perturbation theory in the ratio of low-energy scales to the Wilsonian RG cutoff scale. That Lagrangian is then solved to all orders. In the same way chiral perturbation theory can be used to generate a nonrelativistic, particle-number-conserving Hamiltonian for the nuclear system. This Hamiltonian is valid up to a given order in the chiral expansion and can be employed in any many-nucleon system at low energy. Once written down it should be solved to all orders. The interaction of the nucleus with electromagnetic, weak, and pionic currents can then also be derived in chiral perturbation theory (see, for instance, Refs. [21, 22, 23]). For full consistency the expansion for the interaction with the external probe should be taken to the same chiral order as the maximum order used in the chiral expansion of the NN system Hamiltonian.

An alternative to this approach is the Q-counting scheme pioneered by Kaplan, Savage, and Wise [2, 6]. In this approach power-counting for S-matrix elements is explicit, with all quantities being written as an expansion in the ratio of \sqrt{MB} , p , and m_π to some underlying scale Λ . In practice, the deuteron (or the quasi-bound state in the 1S_0 channel) is generated by iterating a non-derivative contact interaction to all orders. This builds into the theory an explicit low-energy pole; all other interactions (including one-pion exchange) are then treated perturbatively, with a power-counting for the short-distance physics determined by the renormalization group flows of the operators. A number of observables in the two-nucleon system have been calculated using Q-counting, and it appears to work well for many observables at momenta $p \leq m_\pi$. However, as yet, no observable has been calculated where this Q-counting approach with explicit pions works demonstrably better than a similar Q-counting developed for the pionless effective theory. This does raise the question of how well Q-counting actually incorporates the pionic physics of the NN system.

In this paper we adopt Weinberg's power-counting, and make what are essentially chiral perturbation theory expansions of the NN potential and the two-particle-irreducible kernel for electron-deuteron scattering. Our goal is to examine the physics of deuteron electromagnetic form factors within this approach. Our nucleon-nucleon potential has a structure determined from an effective

field theory expansion. Higher-derivative nucleon-nucleon interactions are suppressed by appropriate powers of the characteristic short-distance scale Λ . Accordingly we follow Lepage’s usage [24] and refer to this scheme as Λ -counting. We will demonstrate that this approach to effective field theory can be used to generate deuteron wave functions which are in accord with the wave functions of much more sophisticated, albeit more phenomenological, nucleon-nucleon potential models, for all radii r larger than about $1/m_\pi$. The difference between these two different types of wave function then occurs almost entirely at short distances, i.e. distances less than $1/m_\pi$. So, if the central tenet of effective field theory is indeed correct in the deuteron system these simpler wave functions should be adequate for the calculation of low-energy observables. To put this the other way around, if our wave functions yield predictions for low-energy quantities which differ significantly from those of potential models then this represents an unexpected sensitivity to short-distance dynamics. Such a sensitivity is unexpected precisely because it is contrary to a very basic belief which underpins the success of effective field theory.

Of course, these wave functions are not physical observables. Nevertheless, as will be discussed below, the asymptotic tails of the wave functions, can, in principle, be obtained from experiment [25]. Physical observables dominated by such tails should be well described in our approach. More generally, the EFT framework used here gives the nonrelativistic impulse approximation as the leading nontrivial order for many observables. As is well known, in this approximation for electron-deuteron scattering the wave functions for particular deuteron magnetic sub-states are probed. Therefore we can test the “central tenet” by computing the three electromagnetic form factors of the deuteron using our effective-field-theory-motivated wave functions and comparing them to results from potential model wave functions. This allows us to examine how large the momentum transfer Q must be before one starts probing the physics at scale Λ_χ which was not explicitly included in the development of the deuteron wave function.

Thus, at the heart of this paper is a calculation of electron-deuteron scattering in effective field theory. A calculation of this process in Q -counting recently appeared [26], but until now Λ -counting has not been applied to this reaction except at zero momentum-transfer [27, 28]. Perhaps this is because the results are not very surprising! Indeed, we shall demonstrate that Λ -counting provides a systematic justification for many of the features observed in potential model calculations of the deuteron electromagnetic form factors: the importance of corrections associated with nucleon structure, the unimportance of relativistic corrections at low Q^2 , and the strong suppression of meson-exchange current mechanisms at these momentum transfers. This approach leads to simple power-counting rules to estimate errors for the form factors F_C , F_Q , and F_M , and we will calculate these quantities up to the next-to-leading nontrivial order in our expansion. Moreover, for certain long-distance observables—such as $\langle r^{2n} \rangle$ —the errors can be estimated even more accurately. The reason that this is possible is because such moments are dominated by the tails of the wave function which, as noted above, are physical quantities.

These arguments show that the deuteron electromagnetic form factors can be concretely related to the deuteron wave function, provided that one works only at low orders in the Λ -expansion. So, we would suggest that these form factors at momenta $Q \sim 2m_\pi$ are a good place to test the efficacy of the competing schemes’ reproduction of the physics of the pion range inside the deuterium nucleus. In particular, this may shed some light on whether one-pion exchange can be treated perturbatively in the two-nucleon system. As mentioned above, it is a consequence of Q -counting that one-pion exchange should be sub-leading, with the leading effect provided by the unnaturally-enhanced contact operators that yield a bound deuteron. Cohen and Hansen have shown that such an expansion for the NN amplitude can be obtained from *any* potential $V_{\text{OPE}} + V_{\text{short}}$, provided that $m_\pi R \ll 1$, where R is the range of the short-distance potential V_{short} [29]. While this shows

that one-pion exchange can indeed be treated perturbatively in a particular formal limit it remains unclear whether m_π is small enough in the physical world to make the results thereby obtained useful [30].

Our paper is structured as follows. In Sec. 2 we review the basic physics of nucleon-nucleon potential models, provide a more detailed introduction to effective field theory than the thumbnail sketch given here, and outline our particular approach to the calculation of deuteron wave functions and electromagnetic properties. Section 3 contains the details of our calculation of this wave function, along with some specific error estimates for quantities like $\langle r^{2n} \rangle$. Section 4 gives a description of the expansion that Λ -counting provides for the kernel of the electron-deuteron scattering reaction. In Sec. 5 we present our results from this calculation. These include results for static deuteron properties, which are shown to have errors in accord with our systematic error estimates, and results for F_C , F_Q and F_M . Note that in the latter cases we do not compare with experimental data, since potential model calculations are already very successful for these observables. Rather, by comparing the predictions for F_C , F_Q and F_M with those of potential models we see that the “central tenet” of effective field theory is alive and well in the deuteron. Finally our results are summarized and our conclusions discussed in Section 6.

2 Potential models, effective field theory, and error estimates

Traditionally, low-energy phenomena in light nuclei have been described using the Schrödinger equation with a model for the NN potential. In practice, all sensible potentials include the one-pion exchange potential (OPEP) as their long-range tail. The inclusion of OPEP is required on theoretical grounds, as it uniquely describes the longest-range part of the strong interaction. This can also be seen experimentally, since OPEP is essential in the description of the higher partial waves in NN scattering. On the other hand, apart from their common one-pion exchange tails, different NN potential models look rather dissimilar. Some include a set of single (or, in some cases, double) boson-exchanges to generate the short-distance piece of the interaction, as in the Nijmegen [31, 32, 33] or the original Bonn potentials [34, 35]. Others have a more phenomenological short-distance structure, e.g. the Reid potential [32, 36], the Paris potential [37, 38], or the Argonne potentials [39, 40]³. In all cases the short-distance potential must be rich enough to describe the existing data. Once a particular form is chosen the Schrödinger equation is solved and the free parameters are fit to the NN scattering data and certain deuteron bound state properties.

This is now an extremely sophisticated enterprise. Potentials exist which fit the np and pp scattering data up to laboratory energies of 350 MeV with a χ^2 per degree of freedom close to one [32, 39, 41]. Such models also do a very good job in describing a wide array of low-energy data obtained using electroweak probes. To calculate the response of the nucleus to these probes, single-nucleon currents and potential-model wave functions are often employed. In some cases such a construction does not lead to a conserved electromagnetic current, and so additional “model independent” two-body currents are added to restore current conservation (see, for instance [42]). Conversely, transverse currents are not constrained by current conservation and so are always “model dependent”. The forms chosen reflect dynamical assumptions about the important physical mechanisms, e.g. that pionic, ρ -meson, $\rho\pi\gamma$, $\omega\pi\gamma$, $\Delta\pi\gamma$, and $\Delta\rho\gamma$ meson-exchange currents dominate the transverse currents. For a thorough review of this approach and its application in few-nucleon systems see Ref. [43].

³The CD-Bonn potential [41] nominally involves single-meson exchanges at short distances, but the mass of the scalar-isoscalar meson varies from partial wave to partial wave. Therefore in classifying OBE potentials as we have done here it is not entirely clear which class to place it in.

Thus, at the very least, potential models represent an extremely efficient way to describe the data. On the other hand, one difficulty of the approach is that it is hard to know what is learned about the dynamics from the success of the description. The fact that a variety of potential models with different short-distance dynamics all do well in describing the data suggests that a good description of the data does not necessarily indicate identification of the underlying short-distance processes. However, from an effective-field-theory viewpoint this is entirely to be expected. Indeed, the central tenet of effective field theory is that it is not possible to resolve details of short-distance physics from low-momentum observables. Thus in an EFT framework, attempts to get the “right” short-distance NN potential using low-momentum data are futile. This suggests that the limitation be viewed as an opportunity. If details of short-distance physics did strongly affect low-energy observables it is very unlikely we would ever make any good predictions at low energy. After all, there are an infinite number of choices for the short-distance physics and we are unlikely to stumble on the right one. Potential models have predictive power precisely because the details of the short-distance physics do not matter greatly. Thus, the very success of a variety of NN potential models suggests that the NN system is well suited to an EFT treatment.

Having said this, there are a number of different views on how to implement effective field theory ideas in nuclear physics. That originally proposed by Weinberg [14, 15] and employed by Lepage in Refs. [24, 44] allows one to think straightforwardly about the connection of EFT and potential models. As discussed in the Introduction, in this approach, known as “ Λ -counting”, the focus is on power counting at the level of the Hamiltonian in the NN piece of the hadronic Hilbert space. The effective Lagrangian consists of NN contact interactions together with the standard heavy baryon chiral perturbation theory Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{HB}\chi\text{PT}} + \mathcal{L}_{NN}. \quad (2)$$

At lowest order $\mathcal{L}_{\text{HB}\chi\text{PT}}$ may be written in a form which explicitly shows that at this order the interaction terms involving zero- and one-pion field take the form of a standard nonrelativistic Lagrangian for the interactions of pions, nucleons and photons [26]:

$$\begin{aligned} \mathcal{L}_{\text{HB}\chi\text{PT}} = & N^\dagger \left(iD_0 + \frac{\mathbf{D}^2}{2M} \right) N + \frac{g_A}{2f_\pi} N^\dagger \boldsymbol{\sigma} \cdot (\mathbf{D}(\vec{\tau} \cdot \vec{\pi})) N + \frac{1}{2} [(D_\mu \vec{\pi}^\dagger D^\mu \vec{\pi}) - m_\pi^2 (\vec{\pi}^\dagger \vec{\pi})] + \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2), \end{aligned} \quad (3)$$

where $f_\pi = 93$ MeV, and N and $\vec{\pi}$ are the nucleon and pion isodoublet and isotriplet. Chiral symmetry is not manifest here, as the Lagrangian written in terms of the objects which transform appropriately under $SU(2)_L \times SU(2)_R$ has been expanded out in terms of the more familiar $\vec{\pi}$ to yield Eq. (3). \mathbf{E} and \mathbf{B} are the electric and magnetic fields of the photon, and D_μ is the usual covariant derivative, which ensures that $D_\mu N$ transforms in the same way as N under the group $U(1)_{em}$:

$$D_\mu = \partial_\mu + ieQ_{em}A_\mu + \dots, \quad (4)$$

with Q_{em} the isospin-space matrix:

$$Q_{em} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (5)$$

and A_μ the photon field. The ellipses indicate that additional terms in D_μ involving one or more pion fields are needed if it is to also be a covariant derivative under $SU(2)_L \times SU(2)_R$. These

terms do not play a role in any of our calculations. Meanwhile the S-wave piece of the Lagrangian containing the four-nucleon contact terms is:

$$\mathcal{L}_{NN} = -\frac{1}{2}C_0(N^\dagger N)^2 - \frac{1}{2}C_2[(N^\dagger N)(N^\dagger \mathbf{D}^2 N) + (N^\dagger N)(\mathbf{D}^2 N^\dagger N)] + \dots \quad (6)$$

This Lagrangian can then be used to generate an NN potential, V as follows. In principle, V contains all two-particle-irreducible $NN \rightarrow NN$ graphs. As shown by Weinberg and Ordoñez et al. [14, 15, 27], each of these graphs is of a given order in the chiral expansion, and more complicated graphs occur at higher order. The other issue here is how to count the contact interactions C_{2n} . In the framework of “ Λ -counting” for V , we simply use naive dimensional analysis in this nonrelativistic field theory to estimate:

$$C_{2n} \sim \frac{1}{M\Lambda^{2n+1}}. \quad (7)$$

It follows that at leading order in Λ -counting, the NN potential has two pieces [15]:

$$V(\mathbf{q}) = -\frac{g_A^2}{4f_\pi^2} \frac{\sigma_1 \cdot \mathbf{q} \sigma_2 \cdot \mathbf{q}}{\mathbf{q}^2 + m_\pi^2} (\vec{\tau}_1 \cdot \vec{\tau}_2) + C_0, \quad (8)$$

where C_0 is a contact interaction which is a constant in momentum space (or equivalently, a three-dimensional delta-function in coordinate space). This contact operator gives the short-distance piece of the NN interaction, while one-pion exchange gives the long-range tail. Indeed, at any chiral order an analogous separation into a long- and short-distance NN EFT potential can be made. At low orders in the expansion the short-distance part has straightforward Λ -counting, with the higher-derivative operators always suppressed by powers of p/Λ . Meanwhile, the long-distance piece is given by one-pion exchange plus irreducible two-pion exchange plus three-pion exchange plus etc. etc.⁴ The dimension of the operators which these graphs produce in the NN Hamiltonian turns out to increase by two for each additional pion loop [14, 15, 27]. Consequently, these irreducible multiple pion exchanges are also suppressed by powers of m_π/Λ and p/Λ . Both this long-range part of V and the short-range part coming from the contact operators should then be truncated at the same order in the expansion in p and m_π over Λ .

The resulting potential is then used nonperturbatively in a Schrödinger equation. Of course, this procedure will generate divergences unless the contact terms are regulated. This is achieved by taking some (essentially arbitrary) functional form parameterized by a regulator mass to serve as a regulated coordinate space three-dimensional delta function. Practical implementation of the renormalization program is then quite simple: with the regulator mass fixed, one tunes the coefficients so as to reproduce some specified low-energy observables [44, 45]. In Ref. [27] such a fit to both NN phase shifts up to laboratory energies of 100 MeV and deuteron properties was done with the next-to-next-to-next-to-leading order (NNNLO) Λ -counting interaction. Three different regulator parameters and a Gaussian regulator were employed to render finite the divergences which would otherwise have appeared.

With renormalizable theories, the next step would be to take the regulator mass to infinity. In Refs. [1, 10, 44, 46], it is argued that this should not be done for these effective theories. Rather, the regulator mass is to be set at, or near, the scale of the short-distance physics, Λ . One advantage in doing this is that one then expects that the bare coefficients in the expansion will be of the size given by Eq. (7), whereas if regulator scales were chosen to be very different from Λ the values of the

⁴In practice, three-pion exchange is of a range comparable to that of the “short-distance” potential, but, in the chiral limit it is clear that such a separation into long and short-distance pieces could be made unambiguously.

bare coefficients could be quite different from this naive-dimensional-analysis estimate. Moreover, by keeping the regulator finite and of order Λ one avoids the difficulties discussed in Refs. [47, 48].

This particular treatment of short-distance physics in EFT is actually not very different from that employed in some of the traditional potential models described above. Any functional form given by a potential model with characteristic scale Λ and n parameters could be rewritten as the sum of n terms with each term given by derivatives of some “regulated delta-function”. Thus, the gain achieved in using EFT as compared to traditional potential models is really the ability to treat the long-distance physics and the current operators in a manner consistent with each other and with the short-distance physics by working to a given order in Λ -counting in the system’s Hamiltonian. In particular Λ -counting constrains the number and form of current operators allowable at a given order and mandates the type of pion-loop and pion exchange mechanisms which should be included in the calculation. Such formal consistency is very satisfying. It also confers the practical advantage that errors can be estimated in a reliable way. The key point is that the characteristic size of the physics neglected at each stage of the calculation is known. In particular, in this work we use the EFT approach to analyze uncertainties in calculations of electromagnetic properties of the deuteron.

The errors which arise include uncertainties in our wave functions and uncertainties in the current operators—both of which are due to truncations in the Λ -expansion. In discussing these errors we first focus on the tail of the bound state wave function, which, while not directly observable, can be obtained indirectly from scattering data. This can be seen simply in a spectral representation of the general off-shell NN t-matrix. The t-matrix diverges as one approaches a bound state pole in energy. Then, if there is a bound state $|\phi\rangle$ at $E = -B$ the residue there is directly related to the bound state density-matrix via:

$$\lim_{E \rightarrow -B} (E + B) T(E) = (B + H_0) |\phi\rangle \langle \phi| (B + H_0), \quad (9)$$

where H_0 is the free Hamiltonian. Now, consider both sides of this equation in coordinate space. In both the initial and the final state take the relative distance between the two nucleons to be much larger than the range of the interaction. For positive energies the left-hand side can be expressed in terms of the measured NN phase shifts. This gives the coordinate-space tail of the scattering wave function. An extrapolation to the bound state pole at $E = -B$ may then be made to obtain the tail of the bound state wave function. Thus, for systems with a shallow bound state, accurate scattering data determines the asymptotic tail of the bound state wave function in a model-independent way.

Since our interest here is in long-distance phenomena in the deuteron, it seems sensible to choose our fitting procedure to ensure that the asymptotic deuteron wave function is correct. If some other fitting-procedure were employed, e.g. fitting the coefficients in a Taylor expansion of the T matrix or $1/T$, additional errors will be introduced, since the tail of the deuteron wave function would acquire some errors as part of the extrapolation from finite p to $p = i\sqrt{MB}$. These errors are formally higher-order in the power counting. Indeed, they are very small in Λ -counting, as demonstrated in a calculation without explicit pions by Park et al. in Ref. [8]. There the 3S_1 scattering length and effective range were fitted and A_S was then reproduced to about 1% accuracy. Since we are free to choose among the different possible fitting procedures, and these are all equivalent to the order we work here, we fit to the experimental values for the deuteron binding energy B , and for the asymptotic S and D-state normalizations A_S and A_D . This has the merit that the longest-distance parts of the wave function is described perfectly, within the experimental errors. In particular, the deuteron pole appears at the correct location in the complex energy plane, and has the correct residue.

Proceeding in this way we integrate this essentially-exact asymptotic form from $r = \infty$ inward. Errors become large when we reach one-pion ranges if only the free Schrödinger equation is used to do this integration. If one-pion exchange is added to the differential equation used in integrating inwards the resulting wave functions are accurate down to distances $r \sim 1/m_\pi$. Adding two-pion exchange would further increase the range over which the wave functions are accurate. Moreover, as we shall show in Section 3.4 the chiral suppression of two-pion exchange noted above means that the effects of the two-pion exchange potential will be small even at ranges where they might be thought to be important. Similarly, relativistic corrections to both the nucleon motion and the one-pion exchange potential are suppressed by powers of p/M , where p is the typical momentum inside the bound state. It is therefore quite easy to demonstrate that the wave functions employed are accurate in the region $r > 1/m_\pi$ up to corrections of relative order $(P/\Lambda)^2$, where P could be either the typical nuclear momentum, p , or m_π , and Λ is Λ_χ or M .

Of course, in pursuing the wave function to ever smaller r ultimately one reaches a distance R which is so short that the chiral expansion for the potential fails, and some physics beyond the range of the effective theory starts to play a role. At this point we employ an arbitrary regulator, motivated by the regulated delta-function of Λ -counting, to ensure that the wave function has the correct behavior as $r \rightarrow 0$. The way in which the deuteron wave function is obtained in this “integrating in” picture is very similar to that employed for scattering wave functions in the Nijmegen phase-shift analyses [49, 50].

For probes of the deuteron which probe the scale R the error analysis employed in this paper breaks down completely, since the wave function for $r \leq R$ is largely fallacious. On the other hand, it is one of the central conclusions of this paper that electron-deuteron scattering is *not* such a probe, at least provided that $Q \leq 700$ MeV, and reasonable values for R are chosen.

Thus far we have only spoken about errors in the wave function. Of course, there are corresponding errors in the current. In this work we employ only the next-to-leading (NLO) order chiral-perturbation theory deuteron current. That current is derived in Sec. 4. It represents two particles of charge e and zero and anomalous magnetic moments μ_p and μ_n . No terms corresponding to the nucleon’s charge or magnetic radius appear in the current at this order. Such terms enter as a correction at the next order in the expansion. Other corrections to the one-nucleon current also occur one order beyond that considered here. As for two-body currents, as explained in Sec. 4, in the chiral expansion *all* such effects are suppressed by *at least* two powers of P/Λ relative to the leading-order operator.

If the probe in question does not directly probe the wave function for $r \leq R$, we can write the matrix element of some current operator as:

$$\langle \psi | O | \psi \rangle = (\langle \psi^{(0)} | + \langle \delta \psi |)(O^{(0)} + \delta O)(| \psi^{(0)} \rangle + | \delta \psi \rangle), \quad (10)$$

where the operator O_0 and the wave functions $|\psi^{(0)}\rangle$ are the ones employed in this calculation, and the arguments of the following two sections will show that the relative size of both δO and $|\delta \psi\rangle$ is P^2/Λ^2 . This means that the electron-deuteron observables calculated here should be accurate up to corrections of relative order P^2/Λ^2 , provided that Q is not so large as to probe the short-distance piece of the deuteron wave function. This justifies the nonrelativistic impulse approximation as both the leading and next-to-leading term in this effective field theory expansion of deuteron electromagnetic properties. Furthermore, we will see in Section 4 that the pieces of δO which appear at relative order P^2/Λ^2 and P^3/Λ^3 are those which are known to be important in potential-model calculations of deuteron electromagnetic properties.

3 Wave functions

Here we discuss the wave functions which we use to calculate the deuteron electromagnetic form factors. One thing we aim to test in this work is at what virtual photon momentum the details of the short-range deuteron structure affect the electromagnetic observables F_C , F_Q , and F_M . To this end we think of the wave functions as being generated by integrating the Schrödinger equation in from infinity, given a certain asymptotic tail for the wave function. The parameters of this asymptotic tail can be determined experimentally, at least in the sense of dispersion relations. In the deuteron they are the deuteron binding energy B , the asymptotic S-state normalization, A_S , and the asymptotic D/S ratio, η . (These were also the parameters chosen in Ref. [28].) If no other interaction were included and the asymptotic wave functions were taken to be true all the way into the origin this would be the deuteron of the so-called zero-range approximation, but with the experimental value for A_S (see, for instance [51]). These zero-range wave functions can be improved upon at one-pion ranges if we do not take the asymptotic form all the way in to $r = 0$, but instead include one-pion exchange in the Schrödinger equation when integrating in from $r = \infty$. In this way two different kinds of wave functions, both of which have the incorrect behavior as $r \rightarrow 0$ can be generated, with no assumptions made about the short-distance piece of the NN interaction: “zero-range” wave functions, in which the asymptotic form persists to $r = 0$, and wave functions which are “integrated in” using one-pion exchange to all orders. The details of this procedure are discussed in Section 3.1.

In order to render our calculations finite we must also introduce a short-distance regulator. This procedure is discussed in Sec. 3.2. There, we choose a square well for this purpose, and determine the wave functions for different values of the square-well radius, R . Note that all these wave functions have exactly the same structure outside of the square well, since we use the method of Sec. 3.1 to construct their tails.

Introducing this arbitrary, and undoubtedly incorrect, short-distance potential leads immediately to the question of whether this procedure introduces uncontrollable errors for observables. In Sec. 3.3 we show how, in this potential model, the error introduced by our lack of knowledge of the short-distance piece of the wave function can be estimated for the moments $\langle r^{2n} \rangle$.

In Secs. 3.4 and 3.5 we discuss how to systematically estimate the errors which arise from the inaccuracy of our wave functions *outside* $r = R$. Namely, we ask how the introduction of two-pion exchange, the πNN form factor, and relativistic effects in the deuteron wave function would affect the error bounds derived in Sec. 3.3 for $\langle r^{2n} \rangle$.

Not surprisingly, the size of these errors depends crucially on the radius R at which the short-distance potential starts, so we provide estimates for R in Section 3.6.

3.1 Calculating the wave function by “integrating in”

In this section we explain how the deuteron wave functions

$$|M \mathbf{P}\rangle \tag{11}$$

are calculated.

As explained above, we work in an effective field theory in which nucleons are treated nonrelativistically and relativistic effects are included as a perturbation. Below we will quantify the effect of relativistic corrections. But, for the moment we make the usual nonrelativistic decomposition

of the wave function $|M \mathbf{P}\rangle$:

$$|M \mathbf{P}\rangle = \int \frac{d^3p}{(2\pi)^3} |\mathbf{P} \mathbf{P}\rangle \sum_{LS} \sum_{m_S m_L} \tilde{u}_L(p) (L m_L S m_S | J M) \langle \hat{p} | L m_L \rangle |S m_S\rangle, \quad (12)$$

where $\langle \hat{p} | L m_L \rangle$ are the usual spherical harmonics, $(L m_L S m_S | J M)$ are the Clebsch-Gordon coefficients, \tilde{u}_L is the radial wave function corresponding to orbital angular momentum L , and the spin wave function $|S m_S\rangle$ can be expressed in terms of the single-nucleon spins via:

$$|S m_S\rangle = \sum_{m_1 m_2} (1/2 m_1 1/2 m_2 | S m_S) |1/2 m_1\rangle |1/2 m_2\rangle. \quad (13)$$

The wave functions $\tilde{u}_L(p)$ are the spherical Bessel transforms of the radial wave functions $u_L(r)$,

$$\tilde{u}_L(p) = i^L 4\pi \int dr j_L(pr) r u_L(r), \quad (14)$$

where j_L is a spherical Bessel function.

In the case of the deuteron the spin and total angular momentum of the state are $S = J = 1$, and so the states $L = 0$ and $L = 2$ can contribute to the wave function. Following convention we define $u(r) = u_0(r)$ and $w(r) = u_2(r)$.

Suppose that the asymptotic wave functions $u_0(r)$ and $u_2(r)$ are known. Then we wish to solve the radial Schrödinger equation:

$$-\frac{d^2 u_l}{dr^2} + \frac{l(l+1)}{r^2} u_l(r) + \sum_{l'=0,2} M V_{ll'}(r) u_{l'}(r) = -\gamma^2 u_l(r), \quad (15)$$

where $\gamma^2 = MB$, and we have allowed for the possibility that V is a non-central potential which mixes states of different orbital angular momentum. This equation is to be solved subject to the boundary conditions:

$$u(r) \longrightarrow A_S e^{-\gamma r} \text{ as } r \longrightarrow \infty, \quad (16)$$

$$w(r) \longrightarrow A_D \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right) e^{-\gamma r} \text{ as } r \longrightarrow \infty; \quad (17)$$

where γ and the asymptotic normalizations A_S and A_D are given quantities.

In the pionless theory $V_{ll'}(r) = 0$ and so these asymptotic forms persist all the way into $r = 0$. We define:

$$u^{(0)}(r) = A_S e^{-\gamma r}, \quad (18)$$

$$w^{(0)}(r) = A_D \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right) e^{-\gamma r}; \quad (19)$$

as these so-called “zero-range” solutions.

In the theory with pions we have the following angular-momentum decomposition for the one-pion exchange potential in the deuteron channel:

$$V_{00}(r) = V_C(r) \quad (20)$$

$$V_{02}(r) = V_{20}(r) = 2\sqrt{2}V_T(r) \quad (21)$$

$$V_{22}(r) = V_C(r) - 2V_T(r), \quad (22)$$

where the central and tensor pieces of the potential are, respectively:

$$V_C(r) = -m_\pi f_{\pi NN}^2 \frac{e^{-m_\pi r}}{m_\pi r}, \quad (23)$$

$$V_T(r) = -m_\pi f_{\pi NN}^2 \frac{e^{-m_\pi r}}{m_\pi r} \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2} \right). \quad (24)$$

To leading order in the chiral expansion the πNN coupling is given by the Goldberger-Treiman relation:

$$f_{\pi NN}^2 = \frac{g_A^2 m_\pi^2}{16\pi f_\pi^2}, \quad (25)$$

although we will not be using this exact value in our work.

We label the wave functions u and w found by solving Eq. (15) with this one-pion exchange potential as $u_\pi(r)$ and $w_\pi(r)$. Similar solutions were obtained by Klarsfeld *et al.*, in the early 1980s [52, 53], and similar ideas were used to generate a D-wave wave function given an S-wave wave function, u , by Ericson and Rosa-Clot [54, 55, 56].

3.2 Short-distance regulator

Note that we have dropped the delta-function piece of the central potential in Eq. (23), since we will never integrate all the way into $r = 0$. Instead, for radii r less than some specified distance R we replace one-pion exchange by a central short-distance potential $V_S(r)$. This short-distance interaction subsumes all the details of the physics of the nucleon-nucleon system at short range. It is designed to ensure the correct $r \rightarrow 0$ behavior of u and w , namely:

$$u(r) \sim r; \quad w(r) \sim r^3 \quad \text{as } r \rightarrow 0. \quad (26)$$

Without the presence of $V_S(r)$ u and w do not have the appropriate $r \rightarrow 0$ limits, and divergences in physical quantities may result. Once these divergences are removed quantities which are dominated by long-distance physics should not be sensitive to the details of the short-distance potential.

Thus we are encouraged to choose a form for the short-distance potential which allows us to compute the wave function for $r < R$ analytically. Hence we choose the sum of a square-well and a surface delta-function. The potentials $V_{l'l''}(r)$ retain the form (20)–(24) for $r > R$ in the theory with pions and are zero in the theory without pions. Then, in the S-wave, for $r < R$ we have the structure:

$$V_{00}(r) = V_0 \theta(R - r) + V_1 \delta(r - R), \quad (27)$$

while in the D-wave:

$$V_{22}(r) = V_2 \theta(R - r). \quad (28)$$

In this “integrating in” approach the radius R is given, and the coefficients V_0 , V_1 , and V_2 may then be derived from the conditions of continuity of the wave function and its first derivative at $r = R$, as well as the condition that the wave function has unit norm. This is equivalent to fitting V_0 , V_1 , and V_2 to B , A_S , and η . Such an approach is similar to that employed in Ref. [28]. The main difference is that our short-distance potential is constructed in coordinate space, and has a different angular-momentum structure than that of Park *et al.*

Specifically, the wave function inside $r = R$ is:

$$u_l(r) = C_l r j_l(\alpha_l r), \quad (29)$$

where $\alpha_l^2 = M(V_l - B)$. Since we have integrated the Schrödinger equation with or without one-pion exchange in to $r = R$ the value of u_l at $r = R$ is known. The coefficients C_l are then easily seen to be:

$$C_l = \frac{u_l(R)}{R j_l(\alpha_l R)}. \quad (30)$$

The coefficient V_1 is set by the requirement that it produce the appropriate change in the first derivative of u_0 at $r = R$.

Meanwhile, α_2 is determined by the requirement that the logarithmic derivative of $w(r)$ be continuous, and α_0 by the requirement that the wave function be normalized to unity. We may write:

$$\int_0^R dr u^2(r) = 1 - \int_R^\infty dr u^2(r) - \int_0^\infty dr w^2(r), \quad (31)$$

where the quantity on the right-hand side can be calculated numerically. Since u and w have the form (29) for $r \leq R$ the left-hand side may be calculated analytically and an equation for α_0 obtained and solved.

Observe that if V_S is set to zero, and the equations (15) are solved all the way into the origin, then the wave function violates unitarity (it has norm greater than one) and does not obey Eq. (26). Of course, the first problem can be removed by adding to the probability density from the wave function for $r > 0$ a piece of negative probability density which contributes to the overall probability density only at $r = 0$. One way to implement this consistently is using the energy-dependent pseudo-potential discussed in Ref. [3]. In that case the wave-function normalization condition is no longer simply $\langle \psi | \psi \rangle = 1$, since there will be a contribution to the normalization integral from the energy-dependent potential. When this is evaluated properly it transpires that we still have $F_C(0) = 1$, even though the wave function apparently violates the argument given here and below. However, if we restrict ourselves to the usual framework of non-relativistic quantum mechanics, where probability densities are positive-definite and potentials are energy-independent, we see that both the theory with pions and the theory with pions integrated out have a minimum value of R for which Eq. (31) has a solution. If R is made too small the quantity on the right-hand side becomes negative and so cannot equal the (assumed) positive-definite quantity on the left-hand side. If one-pion exchange is used in integrating the wave function in from infinity then the minimum value of R for which (31) has solutions for the short-distance potential (27) is 1.3 fm. Meanwhile, if one-pion exchange is switched off, the minimum value for which (31) has solutions in our calculation is 2.0 fm. Furthermore, if $R < 1.7$ fm and the wave functions u_0 and w_0 are used on the right-hand side, then there is no solution for R , no matter what short-distance potential is employed (again, assuming positive-definiteness of the probability density). Similarly, if $R < 0.83$ fm and the wave functions u_π and w_π are used on the right-hand side then the left-hand side is negative, and so, again regardless of the short-distance potential employed, no solution for R exists. If we assume, as is normally done in non-relativistic quantum mechanics, that the potentials are energy-independent and the wave functions have norm one, then some potential other than one-pion exchange *must* begin to play a role in the dynamics of the deuteron wave function even at radii $r \sim 1$ fm. Such lower bounds on the range of the non-one-pion exchange piece of the NN potential were derived using similar means by Klarsfeld *et al.* many years ago [53].

Now, given any R which is large enough to allow a solution to Eq. (31) we can obtain a wave function which has the following properties:

- It has the appropriate asymptotic behavior, with values for A_S , A_D and the binding energy taken from experimental data;
- In the theory with pions it corresponds to the wave function for the Schrödinger equation solved with one-pion exchange alone for all r greater than the specified R ;
- It is continuous everywhere;
- It is normalized to one;
- It has the appropriate behavior as $r \rightarrow 0$.

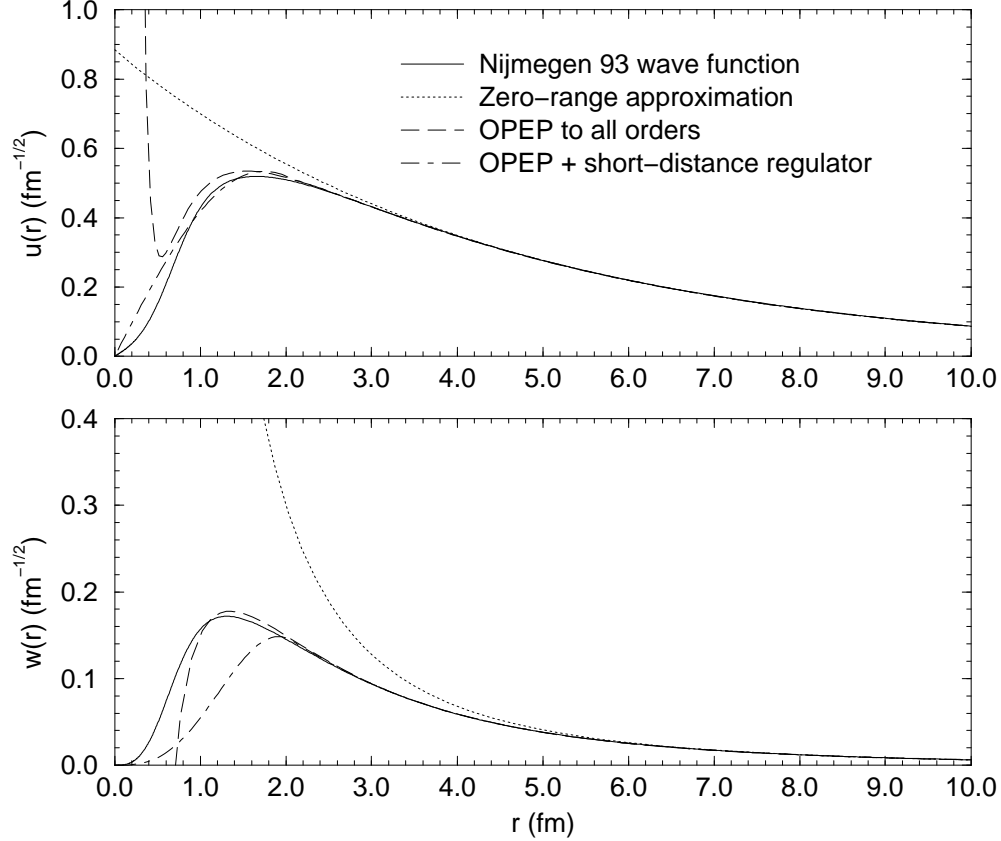


Figure 1: The radial wave functions of the deuteron, $u(r)$ and $w(r)$, for the Nijm93 potential (solid line), as compared to wave functions in which the asymptotic form is integrated in to $r = 0$, both with (dashed) and without (dotted) one-pion exchange. The wave function in which the asymptotic form persists to $r = 0$ is referred to as the “zero-range” solution. The dot-dashed line is the wave function obtained when a short-distance regulator is used in the calculation with one-pion exchange for $r < 2.0$ fm.

The results of this procedure for $R = 2.0$ fm are shown for both u and w , and compared with u_π and w_π and u_0 and w_0 , as well as with the Nijmegen 93 (Nijm93) wave function, in Fig. 1. A_S ,

η , and B were taken to be the central experimental values [57]:

$$B = 2.2246 \text{ MeV}, \quad (32)$$

$$A_S = 0.8845 \text{ fm}^{-1/2}, \quad (33)$$

$$\eta = 0.0253. \quad (34)$$

Meanwhile, the πNN coupling is taken to agree with that of Ref. [32], so that the wave functions constructed here agree (to within experimental error bars) with the Nijm93 wave function in the asymptotic regime and at one-pion ranges:

$$f_{\pi NN}^2 = 0.075. \quad (35)$$

3.3 Corrections to the wave function for $r \leq R$

We will now examine how corrections to the deuteron wave function inside the range of the short-distance potential affect the electromagnetic observables under consideration here. We find, not surprisingly, that the key quantity with which these corrections scale is R .

Consider the observable

$$\langle r^{2n} \rangle \equiv \int_0^\infty dr r^{2n} [u^2(r) + w^2(r)], \quad (36)$$

where u and w are the “true” deuteron wave function⁵. Then, provided that the radius R is not too small the wave function u_π will be accurate for $r \geq R$. Thus the error in $\langle r^{2n} \rangle$, $\delta\langle r^{2n} \rangle$, may be written as:

$$\delta\langle r^{2n} \rangle = \int_0^R dr r^{2n} |u^2(r) + w^2(r) - u_\pi^2(r) - w_\pi^2(r)|. \quad (37)$$

Now it is simple to bound this quantity from above:

$$\delta\langle r^{2n} \rangle \leq \int_0^R dr r^{2n} [u^2(r) + w^2(r)]. \quad (38)$$

Of course, with a “sensible” short-distance potential the error would be much smaller than this, but if the short-distance interaction were a hard core out to radius R , then this would be the error. To estimate the right-hand side we now approximate u and w by a linear and cubic function respectively, yielding:

$$\delta\langle r^{2n} \rangle \lesssim \int_0^R dr r^{2n} \left[u^2(R) \left(\frac{r}{R} \right)^2 + w^2(R) \left(\frac{r}{R} \right)^6 \right]. \quad (39)$$

The quantity $u^2(R)$ may be estimated to be at most of size 2γ , while $w(R)$ has a size of order $2\gamma\eta$, with $\eta = A_D/A_S$ being the asymptotic D/S ratio. Thus the leading correction comes from the first term and takes the form:

$$\delta\langle r^{2n} \rangle \leq \frac{R^{2n}}{2n+3} 2\gamma R. \quad (40)$$

⁵Of course, the wave function is not observable, and may be changed by a unitary transformation (equivalently, by a field redefinition). Nevertheless, once we have specified a particular representation for the operators in the problem the wave function is a well-defined object, and thus we may talk about the “true” wave function.

Naively, we might have expected $\delta\langle r^{2n}\rangle \sim R^{2n}$, but because the deuteron is a shallow bound state, most of the weight of the wave function resides in the tail region, where the dynamics is well known, and so the correction is suppressed by a further factor of γR . Since the typical size of this quantity for the shallow bound state is $(2n)!/\gamma^{2n}$ we see that the fractional error becomes smaller and smaller as n increases, and we give more weight to the tail

$$\frac{\delta\langle r^{2n}\rangle}{\langle r^{2n}\rangle} = \frac{2}{2n+3} \frac{1}{(2n)!} (\gamma R)^{2n+1}. \quad (41)$$

We have made all these arguments for the effective theory with pions. They apply equally well to the one without pions, as long as the radius R is made appropriately larger.

We would like to convert these into estimates for the scale of breakdown of the calculation of F_C , F_Q and F_M . Consider, for instance, the expression for $F_C(Q)$ in the nonrelativistic impulse approximation:

$$F_C(Q) = \int_0^\infty dr j_0\left(\frac{Qr}{2}\right) [u^2(r) + w^2(r)]. \quad (42)$$

This may be expanded in products of powers of Q^2 and even moments of the distribution $u^2(r)$. But that expansion does not converge for $Q \geq 2\gamma$. It does, however, give us an estimate of the Q^2 at which our calculation for F_C will become invalid. Since we already know the behavior of the errors $\delta\langle r^{2n}\rangle$ we find

$$\delta F_C(Q) = \gamma R \sum_{n=1}^\infty c_n \left(\frac{QR}{2}\right)^{2n}, \quad (43)$$

where the c_n 's are dimensionless numbers of order one. Therefore we expect that any description of deuteron form factors based on this approach will cease to be valid when $Q/2 \sim 1/R$. Similar results hold for the form factors F_M and F_Q , although it should be noted that if the deuteron D-state dynamics is not reproduced with moderate accuracy these observables will deviate from the experimental data even at low Q^2 .

3.4 The role of two-pion exchange and the πNN form factor

In the theory with dynamical pions one obvious candidate for R is the range where two-pion exchange begins to play a significant role in the dynamics. We see from the recent Nijmegen phase-shift analysis [50] that this occurs inside $r = 1.8$ fm. However, unlike many short-distance pieces of the NN potential two-pion exchange is constrained by chiral symmetry. It can be consistently treated in the framework of chiral perturbation theory (see, for instance [27, 58]), and it transpires that the coefficients of the irreducible two-pion exchange kernel are suppressed by two chiral orders, relative to the coefficients of one-pion exchange. Thus, we may schematically write the total nucleon-nucleon potential as:

$$V(r) = V_{\text{OPE}}(r) + \left(\frac{P}{\Lambda_\chi}\right)^2 V_{\text{TPE}}(r) + V_S(r). \quad (44)$$

Here $P = m_\pi$ or p , where p is the typical momentum inside the deuteron, which we shall show below is of order m_π itself. The potential V_{TPE} is now nominally the same “size” as V_{OPE} and V_S acts only in the region $0 \leq r \leq R$. The radius R_π is defined as the distance at which the one-pion exchange potential wave functions u_π and w_π begin to differ significantly from the zero-range wave functions u_0 and w_0 . We can now identify three regions in the wave function:

- $r > R_\pi$: Since $\gamma \ll m_\pi$ the wave function is still sizable in this region, but it is well described by the asymptotic form (16)–(17);
- $R_\pi > r > R$: Here the wave functions u_π and w_π are good approximations to the “true” wave function. Two-pion exchange can affect the wave function, but its effects are controlled by the chirally-small parameter P/Λ_χ ;
- $r < R$: Here the wave function is purely the result of a crude potential; the only feature in this region which we know to be accurate is the $r \rightarrow 0$ behavior (26).

We can assess the value of R_π by comparing the zero-range wave functions u_0 and w_0 with those u_π and w_π obtained by integrating in using the one-pion exchange potential. Due to the separation of scales $\gamma \ll m_\pi \ll \Lambda_\chi$ there should be a sizable region where the wave functions in the theory with pions are good approximations to the “true” wave functions and yet differ significantly from the zero-range wave functions. The comparison can be made in Fig. 1 and we conclude that for the pionless theory 2.5 fm is a reasonable value for the radius at which u_π starts to deviate significantly from u_0 . Note that for w the distance at which w_0 and w_π differ significantly is somewhat larger. In both cases the distances in question are significantly larger than the naive estimate $R_\pi \sim 1/m_\pi$.

The plots of Fig. 1 confirm that there is indeed the separation of regions in the wave function required for an effective field theory treatment of the deuteron to make sense. Given such a separation we can identify the correction which two-pion exchange will make to the observables $\langle r^{2n} \rangle$ discussed above. Firstly, we note that in the region $r > R$ we can treat V_{TPE} as a perturbation. Consequently we write:

$$|\psi\rangle = |\psi^{(0)}\rangle + |\delta\psi\rangle, \quad (45)$$

where $|\delta\psi\rangle$ is $O(P^2)$ relative to the wave function $|\psi^{(0)}\rangle$, constructed with u_π and w_π , that was used above in estimating the errors $\delta\langle r^{2n} \rangle$.

Next we note that the two-pion exchange correction to $\langle r^{2n} \rangle$ can only come from the region $R_\pi > r > R$. Thus, if we write the u and w from $|\delta\psi\rangle$ as δu and δw we have:

$$\langle \psi | r^{2n} | \psi \rangle \approx \int_0^\infty dr r^{2n} (u_\pi^2(r) + w_\pi^2(r)) + 2 \int_R^{R_\pi} dr r^{2n} (u_\pi(r) \delta u(r) + w_\pi(r) \delta w(r)) + \delta\langle r^{2n} \rangle, \quad (46)$$

where the first term is the object whose error, $\delta\langle r^{2n} \rangle$, in the region $0 \leq r \leq R$, was assessed above. The second term is an additional error, however this error may be bounded quite straightforwardly. Namely, since V_{TPE} is being treated in perturbation theory the typical size of δu will be $O(P^2)$ relative to u . This suggests:

$$\langle \psi | r^{2n} | \psi \rangle \approx \int_0^\infty dr r^{2n} (u_\pi^2(r) + w_\pi^2(r)) + \frac{2}{2n+3} \gamma \left(R_\pi^{2n+1} - R^{2n+1} \right) \frac{P^2}{\Lambda_\chi^2} + \delta\langle r^{2n} \rangle, \quad (47)$$

where we have made a somewhat simplistic assessment of the size of the extra error by using the bounds derived above on the error if u_π is completely fallacious between $r = R_\pi$ and $r = R$, and then multiplying that bound by the relative size $\delta u/u_\pi$.

In fact, the effects of two-pion exchange graphs will in general be smaller than this, since their range is considerably less than that of one-pion exchange mechanisms, and so additional exponential suppressions will ameliorate the difference $u_\pi - u$ in the region near $r = R_\pi$. In other words, two-pion exchange graphs not only occur at a range which is shorter than that of one-pion exchange, their effects are chirally suppressed relative to one-pion exchange at the same distances by a factor P^2/Λ_χ^2 . This is an argument based on chiral symmetry which lends support to the

conclusion of Ericson and Rosa-Clot that the effect of two-pion exchange on the number extracted for A_D/A_S is only five percent at most, with about half of that coming from resonant two-pion exchange in the rho-meson region [54].

There will also be corrections to one-pion exchange due to the πNN form factor. Any pionic corrections will be of one-pion range, but will also be chirally suppressed, and so will not be dealt with here. Indeed, Kaiser *et al.* [58] claim that the pionic corrections give only mass and wave function renormalization corrections to one-pion exchange up to $O(P^3)$ in the chiral expansion. At whatever order such corrections do first contribute, their effects can be allowed for in the same fashion in which we deal with relativistic corrections to the one-pion exchange below.

Pionic corrections to the πNN vertex are not normally included in NN potentials. One set of corrections which *are* usually included are those due the nucleon's finite size. These are often parametrized by a monopole form factor, leading to a momentum-space one-pion exchange potential

$$V(\mathbf{q}) = -\frac{4\pi f_{\pi NN}^2}{m_\pi^2} \frac{\sigma_1 \cdot \mathbf{q} \sigma_2 \cdot \mathbf{q}}{\mathbf{q}^2 + m_\pi^2} \left(\frac{\Lambda^2 - m_\pi^2}{\Lambda^2 + \mathbf{q}^2} \right)^2 (\vec{\tau}_1 \cdot \vec{\tau}_2). \quad (48)$$

In fact, in coordinate space this potential gives central and tensor potentials for the deuteron channel [54]:

$$V_C(r) = -m_\pi f_{\pi NN}^2 \left\{ \frac{e^{-m_\pi r} - e^{-\Lambda r}}{m_\pi r} - \frac{\Lambda^2 - m_\pi^2}{\Lambda m_\pi} e^{-\Lambda r} \right\}; \quad (49)$$

$$V_T(r) = -m_\pi f_{\pi NN}^2 \left\{ \left[\frac{1}{m_\pi r} + \frac{3}{(m_\pi r)^2} + \frac{3}{(m_\pi r)^3} \right] e^{-m_\pi r} - \left[\frac{3}{(m_\pi r)^3} + \frac{3\Lambda}{m_\pi^3 r^2} + \frac{\Lambda^2}{m_\pi^3 r} \right] e^{-\Lambda r} - \frac{1}{2} \left(\frac{\Lambda^2}{m_\pi^2} - 1 \right) \left(\frac{1}{m_\pi r} + \frac{\Lambda}{m_\pi} \right) e^{-\Lambda r} \right\}. \quad (50)$$

These potentials are exactly the same as Eqs. (23) and (24) to a radius $r \sim 1/\Lambda$. Since Λ is typically of order 1 GeV, these corrections to one-pion exchange are almost irrelevant if OPEP is only used for radii $r \geq 1.5$ fm. The effects of the πNN form factor appear in the Λ -counting NN potential as higher-order corrections to (8) [27].

3.5 Relativistic corrections to the nucleon motion

Relativistic corrections to both the potential V and the free-nucleon propagation should also be considered. The latter have no definite range. They are, however, suppressed by a factor:

$$\frac{\langle p^4 \rangle}{8M^3} \frac{1}{\gamma}, \quad (51)$$

and so we will make only corrections of this relative order to the $\langle r^{2n} \rangle$'s.

Relativistic corrections to the one-pion exchange potential can be dealt with consistently in the chiral expansion. The leading effects have the same chiral order as two-pion exchange. (See Refs. [27, 58] for details.) Of course, in this case the suppression relative to the leading one-pion exchange mechanism is due to the smallness of the parameter p/M , rather than the (larger) chiral parameter m_π/Λ_χ . We see that corrections due to relativistic pieces of one-pion exchange will make a contribution to $\langle r^{2n} \rangle$:

$$\delta \langle r^{2n} \rangle_{\text{ROPE}} \approx \frac{2}{2n+3} \gamma \left(R_\pi^{2n} - R^{2n} \right) \frac{\langle p^2 \rangle}{M^2}. \quad (52)$$

The key question is therefore, what is the size of the parameter $\delta^2 \equiv \langle p^2 \rangle / M^2$? In the pionless theory the answer to this question is straightforward. The dominant contribution comes from the piece of the wave function inside the short-distance potential. Hence the precise value of this operator is not well predicted in the effective theory. Nevertheless, it is easy to see that the quantity δ is indeed generically small. Straightforward evaluation of p^2 with the wave function in the pionless theory gives:

$$\begin{aligned} \langle p^2 \rangle &\approx \alpha_0^2 \left(1 - \frac{A_S^2}{2\gamma} e^{-2\gamma R_\pi} \right) \\ &\approx 2\alpha_0\gamma (\alpha_0 R_\pi). \end{aligned} \quad (53)$$

To avoid confusion we have denoted by R_π the range to which the short-distance potential extends in the *pionless* theory. But typically $\alpha_0 \sim 2/R_\pi$, so,

$$\langle p^2 \rangle \approx 8 \frac{\gamma}{R_\pi}, \quad (54)$$

which, for a generic value, $R_\pi = 3$ fm, is much less than M^2 , thus ensuring a small value of δ in the pionless theory.

In the theory with explicit pions we argue that the dominant contribution again comes from the short-distance piece of the potential. Thus in essence the only thing that changes for the above argument when pions are added is that R_π becomes R . The shortest R we will consider, $R = 1.5$ fm, which still only gives $\langle p^2 \rangle^{1/2} \approx 220$ MeV.

Therefore although $\langle p^2 \rangle = MB$ does not hold, and this quantum average is not well described in our effective field theory approach to the deuteron, we may confidently say that relativistic corrections to the deuteron electromagnetic current are suppressed. The quantitative statement is that

$$\frac{\langle p^{2n} \rangle}{M^{2n}} \approx 4 \left(\frac{2}{MR} \right)^{2n-1} \frac{\gamma}{M}. \quad (55)$$

Note again that there is a crucial additional suppression by a factor γ/M due to the dominance of the tail region in the shallowly-bound deuteron state.

3.6 Estimates of R from potential models

We now discuss typical values of the scale R . The Nijmegen phase shift analyses of Refs. [49, 50] both suggest that inside $r = 1.4$ fm some dynamics other than two-pion exchange starts to play a significant role⁶. Potential models based on one-boson exchange interactions also have significant strength at $r \approx 1.5$ fm which does not come from either one-pion exchange or the two-pion exchange graphs in which the exchanged pions do not interact. Effects such as “correlated two-pion exchange” are thought to provide strength in this region and are parametrized in the one-boson exchange interaction via (possibly broad) σ and ρ mesons. The masses of these mesons are generally of order 500–800 MeV. Although their full effect will not be felt until distances less than 0.5 fm are reached they will already have some impact on the wave function at $r = 1.5$ fm. The low-momentum effects of these heavier mesons will appear in the effective theory, but it is not yet clear whether this will allow a good description of the deuteron wave function at, say $r = 1$ fm. Certainly, given the low-order NN interaction used here, for this work taking $R \geq 1.5$ fm is sensible. Whether this

⁶As discussed previously, the requirement that the wave functions u_π and w_π result in a normalizable $\psi(\mathbf{r})$ leads to a lower-bound on R once a particular short-distance potential is chosen. Here that bound is 1.3 fm. In the phase shift analysis of Ref. [49] a similar bound, $R \geq 1.4$ fm, was found.

distance can really be thought of as being due to the hadronic scale Λ_χ is a delicate matter which will engender some debate, since there is a factor of three or four between $1/\Lambda_\chi$ and 1.5 fm. In particular, Cohen and Hansen [29, 59] have suggested that while an expansion in m_π/Λ_χ is likely to be reasonable, an expansion in $m_\pi R$ is questionable if R is this large.

Finally, we can look at the plots in Fig. 1, where we compare the results for the wave function $u_\pi(r)$ with those from a typical phenomenological NN potential, the Nijm93. We see that for a range of R 's, from $R = 1.5$ fm to $R = 2.5$ fm, the wave function u and w will be generically close to those from the much more sophisticated Nijm93. This agreement persists all the way down to radii $r \approx 0.5$ fm.

For a number of reasons then, it seems that $R = 1.5$ fm is a sensible range to choose for the onset of short-distance physics in the NN interaction which is beyond the scope of the effective theory. However, we observe that, at least for the short-distance potential used here, such a choice of R leads to wave functions which track closely with the “true” wave function to values of r of order 0.5 fm. Thus, it may well be that the errors obtained when the wave functions from such models are used will be significantly smaller than the estimates derived above, since the range over which the error in $\langle r^{2n} \rangle$ accrues is actually significantly less than R .

4 The electromagnetic current

Thus far our approach is not different to a primitive potential model calculation of the deuteron wave function. Our deuteron wave function has a tail which is constrained by the experimental data, a section at one-pion range calculated from the one-pion exchange potential, and some short-distance piece inside $r = R$, which here is produced by a potential that is completely phenomenological. This is very similar to the procedure adopted in the Nijmegen phase shift analyses of Refs. [49, 50]. We are arguing that such a calculation is in the spirit of effective field theory since we have separated the effects that different scales in the problem, γ , m_π , and R have on the wave function. We now wish to demonstrate that we may make truly systematic statements about observables. In this section we will first explain how standard techniques of effective field theory allow us to precisely quantify the validity of the nonrelativistic impulse approximation for the calculation of the deuteron electromagnetic form factors. The goal is to show how the effect of the known scales in the problem can be traced through so that the impact of these scales on corrections to the calculations of measured quantities can be known, even if the corrections themselves are not calculated.

This allows us to calculate the electromagnetic current for the deuteron, order-by-order in an expansion in p , Q and m_π . As we shall see, it is a straightforward matter to make such an expansion of the full single-nucleon currents. It is less straightforward to obtain a systematic expansion for the two-body currents which contribute to electron-deuteron scattering. To do that we use powerful effective field theory techniques to quantify the order in the chiral expansion at which such corrections appear. These techniques amount to using the constraints of the symmetries in the problem (especially chiral symmetry) to determine which operators can appear in the two-body piece of the current. This operator will then have a definite dimensionality. As we will explain below, the assumption that the coefficient of this operator is natural gives a direct estimate of the effect of the two-body mechanisms. The result is an expansion for the electron- NN scattering kernel based on naive dimensional analysis, or Λ -counting, as developed by Weinberg [14, 15, 16, 24, 44].

4.1 Definitions

In this section we discuss the calculation of the electromagnetic form factors of the deuteron. Here we will work always in the Breit frame, where the kinematics are as displayed in Fig. 2. We must calculate the matrix elements

$$\langle M' \mathbf{Q}/2 | J_\mu | M - \mathbf{Q}/2 \rangle \quad (56)$$

of the deuteron electromagnetic current to which a virtual photon of three-momentum $\mathbf{Q} = Q\hat{z}$ couples, when the deuteron is in states with specific magnetic quantum numbers M and M' . In fact, due to symmetry, only three of these matrix elements are independent [60]. Thus, the electromagnetic structure of the deuteron may be parametrized in terms of three form factors, $F_C(Q^2)$, $F_M(Q^2)$, and $F_Q(Q^2)$, which are related to the Breit frame matrix elements via:

$$F_C(Q^2) = \frac{1}{3e\eta_1} [\langle 0 \mathbf{Q}/2 | J_0 | 0 - \mathbf{Q}/2 \rangle + \langle 1 \mathbf{Q}/2 | J_0 | 1 - \mathbf{Q}/2 \rangle + \langle -1 \mathbf{Q}/2 | J_0 | -1 - \mathbf{Q}/2 \rangle], \quad (57)$$

$$F_Q(Q^2) = \frac{2}{eQ^2\eta_1} [\langle 0 \mathbf{Q}/2 | J_0 | 0 - \mathbf{Q}/2 \rangle - \langle 1 \mathbf{Q}/2 | J_0 | 1 - \mathbf{Q}/2 \rangle], \quad (58)$$

$$F_M(Q^2) = \frac{\sqrt{2}M_d}{eQ\eta_1} \langle +1 \mathbf{Q}/2 | J^+ | 0 - \mathbf{Q}/2 \rangle; \quad (59)$$

where $\eta_1 = \sqrt{1 + \frac{Q^2}{4M_d^2}}$, and $J^+ = J^1 + iJ^2$. At $Q^2 = 0$ the form factors are normalized to:

$$F_C(0) = 1; \quad (60)$$

$$F_Q(0) = Q_d; \quad (61)$$

$$F_M(0) = \mu_d \frac{M_d}{M}; \quad (62)$$

where Q_d is the deuteron quadrupole moment, and μ_d is the magnetic moment of the deuteron in units of nuclear magnetons.

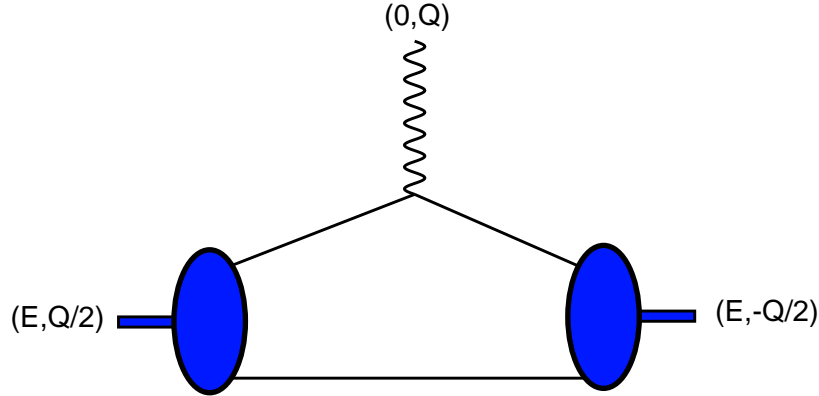


Figure 2: The kinematics for electron-deuteron scattering in the Breit frame. Only the impulse approximation diagram is shown.

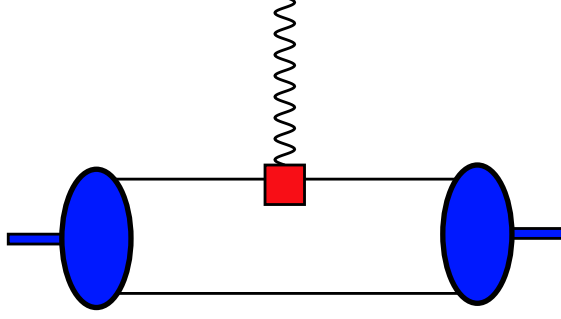


Figure 3: A generic one-body current contribution to electron-deuteron scattering. Here the square indicates any γNN vertex, and the blobs are deuteron vertex functions.

In general, the current to which the electron couples in electron-deuteron scattering may be expressed as:

$$\langle \mathbf{P}' \mathbf{p}' | J_\mu | \mathbf{P} \mathbf{p} \rangle = (2\pi)^3 j_\mu^{(1)}(\mathbf{p}, \mathbf{Q}) \delta(\mathbf{p}' - \mathbf{p} - \mathbf{Q}/2) + j_\mu^{(2)}(\mathbf{p}, \mathbf{p}'; \mathbf{Q}), \quad (63)$$

where \mathbf{p} and \mathbf{p}' are the initial and final relative momenta in the deuteron state, and $\mathbf{Q} = \mathbf{P}' - \mathbf{P}$ is the three-momentum of the photon exchanged between the electron and the deuteron. The one and two-body currents discussed here are depicted in Figs. 3 and 4.

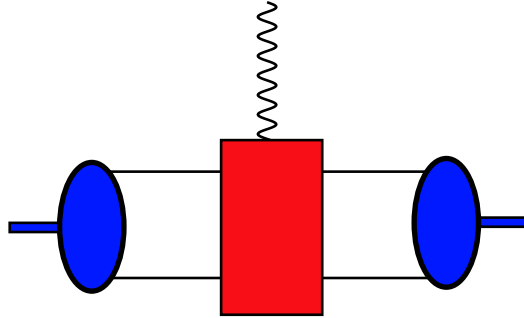


Figure 4: The two-body current contribution to electron-deuteron scattering. The rectangle is the two-body $NN \rightarrow NN\gamma$ kernel, which in this work is discussed in the framework of chiral perturbation theory. The blobs are deuteron vertex functions.

4.2 One-body currents

For on-shell nucleons the most general form that the single-nucleon current, $j_\mu^{(1)}$, can take, consistent with Lorentz invariance, time-reversal invariance, and current conservation, is:

$$j_\mu^{(1)}(\mathbf{p}, \mathbf{Q}) = e_N \bar{u}(\mathbf{p} + 3\mathbf{Q}/4) \left[F_1(Q^2) \gamma_\mu + F_2(Q^2) \frac{i}{2M} \sigma_{\mu\nu} Q^\nu \right] u(\mathbf{p} - \mathbf{Q}/4), \quad (64)$$

where e_N is the nucleonic charge. Note that we have labeled the momenta as they would occur were we to calculate electron-deuteron scattering in the Breit frame and then look at the single-nucleon current as a function of the relative momentum in the deuteron, \mathbf{p} . For the deuteron case, F_1 and F_2 are isoscalar combinations of the proton and neutron Dirac and Pauli form factors:

$$F_1(Q^2) = F_1^{(p)}(Q^2) + F_1^{(n)}(Q^2); \quad (65)$$

$$F_2(Q^2) = F_2^{(p)}(Q^2) + F_2^{(n)}(Q^2). \quad (66)$$

Now we expand $j_\mu^{(1)}$ in powers of the momenta \mathbf{p} and \mathbf{Q} . This involves expanding out both the Dirac structures which arise from Eq. (64), and the single-nucleon form factors $F_1(Q^2)$ and $F_2(Q^2)$. The first expansion is governed by ratios of momenta to the nucleon mass, which we denote generically by the mnemonic:

$$\delta^2 \sim \frac{p^2}{M^2}, \frac{p \cdot Q}{M^2}, \frac{Q^2}{M^2}. \quad (67)$$

Corrections at order δ^2 include the usual spin-orbit and Dirac-Foldy relativistic corrections to the charge operator [61]. Meanwhile, the second expansion will be governed by the parameter Q^2/Λ_χ^2 . It is not surprising that when working to one order beyond the lowest nontrivial contributions, the current is simply due to two objects of charges one and zero, and to anomalous magnetic moments μ_p and μ_n . Corrections due to the neutron and proton radii appear at the next order:

$$j_0^{(1)}(\mathbf{p}, \mathbf{Q}) = e \left(1 + O(\delta^2) + O\left(\frac{Q^2}{\Lambda_\chi^2}\right) \right), \quad (68)$$

$$j_+^{(1)}(\mathbf{p}, \mathbf{Q}) = e \left(\frac{p_+}{M} + \frac{Q}{2M}(\mu_p + \mu_n + 1)\sigma_+ + O(\delta^3) + O\left(\delta\frac{Q^2}{\Lambda_\chi^2}\right) \right). \quad (69)$$

It should be noted that due to reparametrization invariance any nonrelativistic chiral Lagrangian must reproduce exactly the corrections in δ found by expanding out Eq. (64). In such an approach the corrections in Q^2/Λ_χ^2 will appear as higher-derivative operators of the type $N^\dagger \nabla^2 A_\mu N$. These operators have unknown coefficients which must be fit from the experimental data. The reader who wishes to see an analysis of the single-nucleon current carried out in this completely proper fashion should refer to Ref. [62]. Note that examination of the proton form factor data suggests that the expansion of the single-nucleon form factors in powers of Q will break down completely at a scale Q of order 800 MeV, and will begin to fail well before that.

We will take the expressions for j_0 and j_+ only at the leading non-vanishing order in each case. The corrections to these expressions are then suppressed by at least two powers of δ . Above we showed that $\langle p^2 \rangle/M^2$ was a generically small quantity for the wave functions discussed here, and therefore we can be confident that these NNLO corrections are indeed higher-order effects provided that $Q < 1$ GeV.

4.3 Two-body currents

We are now faced with the task of quantifying the effect of two-body currents on the observables under consideration. It is here that the chiral counting approach advocated by Weinberg [14, 15, 16], allows us to make real headway. For a review of this approach see [18]. Here we just restate and employ the results which we need to proceed.

The contribution of two two-body pionic mechanisms to the deuteron current is depicted in Fig. 5. It is straightforward to count the powers of momentum or pion mass in such graphs. If we

generically label internal and external momenta and the pion mass as P , i.e. write $p, Q, m_\pi \sim P$, we then have the following rules for the scaling of a given contribution to the electron-deuteron scattering kernel:

- Each two-nucleon propagator scales like $1/P^2$;
- Each nonrelativistic loop integral contributes a factor of P^3 ;
- Each pion propagator scales like $1/P^2$;
- Each vertex contributes *at least* one power of P^0 .

The key point here is the last one. It arises because the spontaneously broken chiral symmetry of QCD implies that pions couple derivatively to the nucleon. Thus the current contributions in Fig. 5 scale *at least* as P^2 .⁷ Therefore we expect that such meson-exchange currents will be of order

$$\frac{m_\pi^2}{\Lambda_\chi^2}, \frac{\langle p^2 \rangle}{\Lambda_\chi^2}, \frac{Q^2}{\Lambda_\chi^2}, \frac{\langle p \rangle Q}{\Lambda_\chi^2}. \quad (70)$$

This is to be compared to the contribution in Fig. 3 when the vertex is taken to be the lowest-order γNN interaction. In that case Fig. 3 has one less loop, one less propagator and one less vertex than Fig. 5, and so naively scales as P^0 . In fact, Eq. (69) suggests that magnetic coupling is a sub-leading effect in this expansion; and therefore if j_+ is used at the γNN vertex in Fig. 3 then the overall graph is order P .

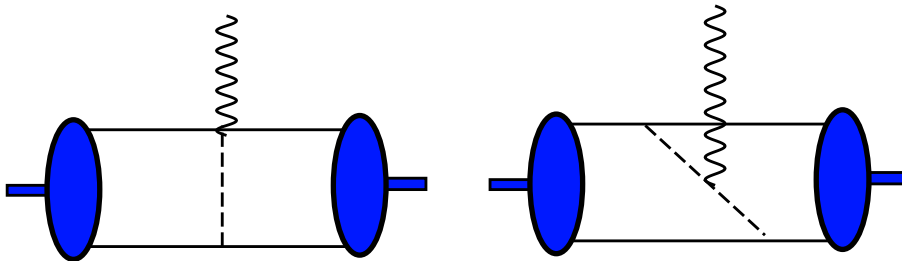


Figure 5: The contribution to electron-deuteron scattering from the leading-order pionic current. Here all vertices are from the leading-order chiral perturbation theory Lagrangian. The graphs with nucleons one and two interchanged also contribute.

This might lead one to think that meson-exchange currents are an NLO effect in the deuteron form factor F_M , but this is not the case. In fact, although the leading effect of meson-exchange currents is nominally $O(P^2)$, the meson-exchange current which occurs at that order is isovector in character and thus vanishes in the deuteron. It is well known that this $O(P^2)$ isovector current is the leading MEC correction to processes such as electrodisintegration of the deuteron (see, for instance, Ref. [42]). In practice this results in the leading effect from meson-exchange currents coming at $O(P^3)$, or NNNLO in the deuteron electric form factor and NNLO in the magnetic form factor. In such an $O(P^3)$ exchange current the $\gamma\pi$ contact interaction with the nucleon has a coefficient that is related to known constants by the requirements of gauge invariance and reparametrization

⁷Here we are counting the charge e as of order P in order to simplify matters.

invariance. We note that this correction to the charge operator is phenomenologically important in potential-model calculations of F_C [43].

At some order above $O(P^2)$ operators enter whose coefficient is not constrained by symmetries, but instead is sensitive to short-distance physics. One such mechanism will be the analogue of the $\rho\pi\gamma$ MEC employed in some hadronic model calculations of electron-deuteron scattering.

Finally, in considering meson-exchange current corrections to F_Q it is important to note that the natural size of F_Q at $Q = 0$ is much smaller than that of F_C and F_M . A natural expectation for $F_Q|_{Q=0} \equiv Q_d$ is:

$$Q_d \sim \frac{\eta}{\gamma^2}. \quad (71)$$

Here η —the asymptotic D-to-S normalization—enters because an S-wave deuteron cannot emit a quadrupole photon via any one-body current mechanism. Thus the power-counting for the contribution of one-body currents to F_Q is exactly the same as for similar contributions to F_C . However, two-body currents can connect the S-wave pieces of the deuteron initial- and final-state wave functions one to another and still result in the emission of a quadrupole photon [12]. The first such effect does not occur until $O(P^5)$ in Λ -counting. On the other hand, crucially it contains no powers of η . Hence, numerically it can be important at much lower order than naively expected. A quick check shows that the contribution of this operator to Q_d will be of order

$$\Delta Q_d \sim \frac{\langle p^3 \rangle}{M\Lambda^2} \frac{1}{\Lambda^2}. \quad (72)$$

Although this is formally of order P^5 , numerically it is as important as NNLO corrections to Q_d .

In summary, our expansion in powers of P for the one- and two-body currents which contribute to electron-deuteron scattering leads us to conclude that the usual nonrelativistic one-body currents for pointlike nucleons will give the correct result up to corrections of NNLO in both F_C and F_M . At that order relativistic effects will correct the one-body current and effects due to electromagnetic form factors of the nucleon will arise. Meson-exchange currents may also contribute to F_M at that order. The first meson-exchange contribution to F_C occurs at NNNLO.

5 Results

From Sec. 3 we have a variety of deuteron wave functions with different short-range behavior. From Sec. 4 we have an understanding of the deuteron current in a systematic Λ -expansion. Therefore in this section we proceed to calculate the electromagnetic form factors of the deuteron. This is done for various values of R ranging from 1.5 fm to 3.5 fm in the theory with pions, and for the minimum R ($R = 2.0$ fm) in the pionless theory. These results are compared with results from a sophisticated potential model, the Nijm93 potential. In every case we employ the nonrelativistic impulse approximation with point nucleons, which, as explained above, represents a calculation to NLO in the Λ -counting electron-deuteron scattering kernel. However, before calculating these form factors we spend some time discussing our results for static properties of the deuteron.

5.1 Static moments

The values of the form factors F_M and F_Q at $Q^2 = 0$ are related to the magnetic and quadrupole moments of the deuteron. It is well known that to this order in Λ -counting these may be obtained

directly as simple integrals of the deuteron radial wave functions u and w :

$$\mu_d = \mu_p + \mu_n - \frac{3}{2}(\mu_p + \mu_n - \frac{1}{2})P_D; \quad (73)$$

$$Q_d = \frac{1}{\sqrt{50}} \int_0^\infty dr r^2 u(r)w(r) - \frac{1}{20} \int_0^\infty dr r^2 w^2(r) \quad (74)$$

with the D-state probability P_D given by:

$$P_D = \int_0^\infty dr w^2(r). \quad (75)$$

Employing the wave functions derived above we obtain the results shown in Table 5.1.

Theory	Value of R (fm)	μ_d (μ_N)	Q_d (fm ²)
Nijm93	N/A	0.847	0.271
Pionless	2.0	0.804	0.322
Pionful	1.5	0.851	0.269
Pionful	2.0	0.858	0.265
Pionful	2.5	0.863	0.258
Pionful	3.5	0.871	0.238

Table 1: Static moments of the deuteron for various values of R in the pionless and pionful theories.

In Table 5.1 we clearly see that the deuteron D-state is much too large in the only deuteron calculated in the pionless theory, while it is too small in all the calculations where pions were included explicitly in the potential—albeit only marginally so for $R = 1.5$ fm. Examination of the deuteron wave functions shows that when integrating in from infinite r the tensor piece of one-pion exchange causes the deuteron D-state wave function to turn over. Without this effect in the calculation the deuteron D-state becomes too large. Once this effect of one-pion exchange is included making R large causes the D-state wave function to turn over too early, thereby leading to a reduction of the D-state probability. Finally, we note that if R is made very large the pionful calculation begins to resemble the pionless calculation, since the one-pion exchange potential is too weak in the region $r > R$ to have much effect on the wave function. These observations echo the work of Ericson and Rosa-Clot, who attempted to use methods like the ones we are employing here to put constraints on η from the experimental value of Q_d [55, 56].

This significant variation in Q_d shows that there is some sensitivity to the short-distance physics in this observable. Indeed, it should be noted that the experimental value for this quantity is actually $Q_d = 0.2859(3)\text{fm}^2$ [63], rather than the value $Q_d = 0.271\text{fm}^2$ obtained using the nonrelativistic impulse approximation and the Nijm93 wave function. Apparently there are significant contributions to this quantity from physics beyond the nonrelativistic impulse approximation. Corrections to this approximation appear at higher orders in the Λ -expansion of the kernel than we have considered here. One correction is a two-body current resulting in the emission of a quadrupole photon, the Lagrangian for which was given in Ref. [12]. This mechanism is of much higher order than the others we have considered in this work, however it still gives approximately as large a contribution to Q_d as terms one order beyond those we have considered here. The importance of this higher-order correction to Q_d is entirely due to the smallness of the leading-order contribution to this observable. In contrast, we shall see below that F_C and F_M , which are not unusually small

quantities, really are insensitive to details of the short-distance physics of the NN potential at radii $r < 2.5$ fm, provided that Q is less than about 700 MeV. But before launching that investigation, we discuss the static moments $\langle r^2 \rangle$ and $\langle r^4 \rangle$ which are dominated by the long-distance piece of the deuteron wave function.

A conservative prediction for the error in these quantities as a function of R was made in the previous section. In Table 5.1 we display results for these quantities for different values of R in both the pionless and pionful calculations. These moments are to be compared to those obtained with the Nijm93 deuteron wave function. The behavior of the moments is clearest in the pionless theory. As expected, the error in $\langle r^4 \rangle$ is smaller, and grows more rapidly with R than in $\langle r^2 \rangle$. For all but the extreme case of the pionless theory with $R \geq 5$ fm the error is on the order of a few percent for $\langle r^2 \rangle$, and is less than one percent for $\langle r^4 \rangle$. Thus there is good agreement between these quantities as calculated with our EFT-motivated wave functions and with the more sophisticated Nijm93 wave function. The reason for this is clear: these are tail-dominated observables, and we have made sure that the tail of all of our wave functions agrees almost exactly with that of the potential-model wave function.

Theory	Value of R (fm)	$\langle r_m^2 \rangle$ (fm ²)	$\langle r_m^4 \rangle$ (fm ⁴)
Nijm93	N/A	3.865	55.35
Pionful	1.5	3.850	54.85
Pionful	2.0	3.848	54.85
Pionful	2.5	3.859	54.87
Pionful	3.5	3.956	55.11
Pionless	2.0	3.940	55.08
Pionless	2.5	3.919	55.06
Pionless	3.5	3.980	55.23
Pionless	5.0	4.384	57.15

Table 2: Values of the matter radii expectation values $\langle r_m^2 \rangle$ and $\langle r_m^4 \rangle$ in the pionless and pionful theories for various values of R . Note that if r is the relative coordinate between the two nucleons then $r_m = r/2$.

5.2 Electromagnetic deuteron form factors

These results give us confidence that in the region where a moment expansion of the deuteron electromagnetic form factors is valid these form factors will be well reproduced by EFT calculations. Furthermore, it means that within the domain of validity of that expansion the error in the form factor can be estimated within our calculational scheme. However, both of these results are of limited use, since an expansion of the electromagnetic form factors of the deuteron in moments of the charge distribution is only valid for $Q^2 \leq 4/\langle r^2 \rangle$; i.e., Q less than of order γ .

Of course, since the wave functions found by integrating in agree almost exactly with the Nijm93 wave function for distances r greater than R we expect the electromagnetic form factors of the deuteron obtained with these wave functions to be in good agreement with those found with the Nijm93 wave function up to Q of order $2/R$. This is a much higher scale than γ ; thus the breakdown of the moment expansion does not mean that an effective field theory description of the deuteron is invalid.

Turning first to the charge form factor of the deuteron we display results in the pionful theory in Fig. 6. These are compared with results from the Nijm93 wave function. The agreement is good out to Q of around 700 MeV, for all but the case $R = 3.5$ fm. As discussed above, that wave function is in fact very close to the one calculated without explicit pions at the same R , so it cannot really be considered a calculation “with pions”. Thus, in practice we do a little better than $Q \sim 2/R$ in the calculations with explicit one-pion exchange. The scale of the breakdown is, for sensible R , more like $Q \sim m_\rho$.

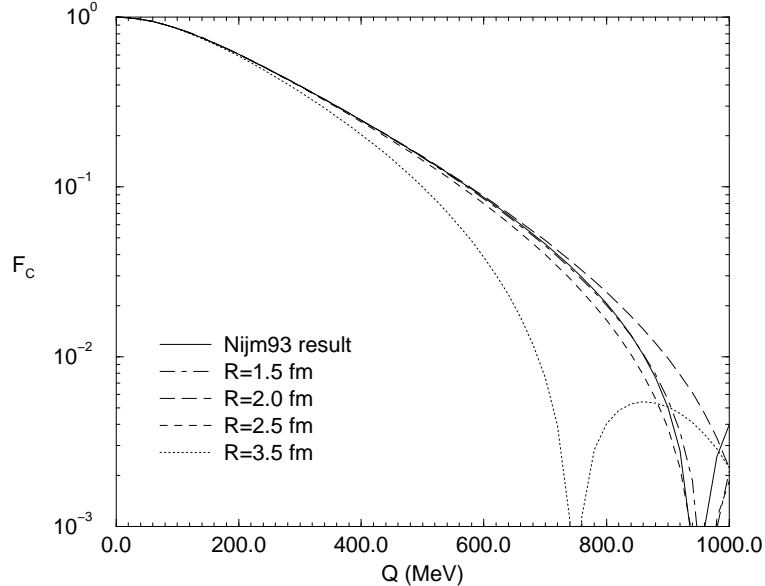


Figure 6: The charge form factor of the deuteron for several different wave functions, all of which have the same tail, as well as the same one-pion exchange part. The solid line is the result for the Nijm93 potential, while our wave functions with different R 's are represented by $R=1.5$ fm (dot-dashed), $R=2.0$ fm (long dashed), $R=2.5$ fm (short dashed), and $R=3.5$ fm (dotted).

What is the scale of the breakdown of the pionless theory? To establish this, in Fig. 7 we present results for F_C from the pionless and pionful theories with $R = 2.0$ fm (the minimum possible R in the pionless case), and also for the Nijm93 potential. From this plot it is clear that the pionless calculation breaks down a little above $Q = m_\pi$, as we would expect. When included as we have done here, one-pion exchange makes a dramatic difference to the range over which the description of the deuteron charge form factor is successful.

Results are presented for the deuteron magnetic form factor F_M in Figs. 8 and 9. These plots lead to conclusions similar to those inferred from the plots of the charge form factor. This is not particularly surprising, given that it was shown in Ref. [26] that to leading order in the Q -expansion the magnetic form factor is directly proportional to the charge form factor. Of course, such a result is basically a consequence of the dominance of the S-wave pieces of the wave function in determining the magnetic structure of the deuteron at low momenta.

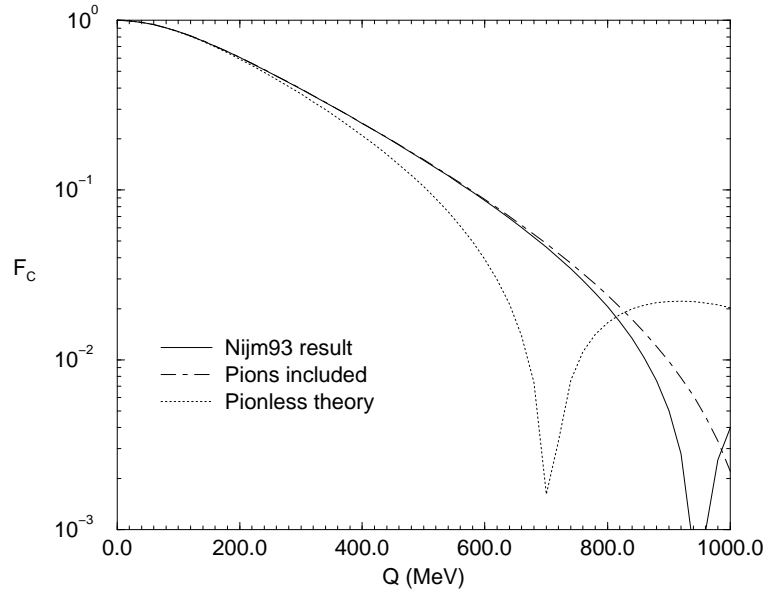


Figure 7: The charge form factor of the deuteron for a short-distance potential of $R = 2.0$ fm, both with and without one-pion exchange included. The solid line is the Nijm93 result, the dotted line is the pionless calculation, and the dot-dashed line is a calculation including one-pion exchange.

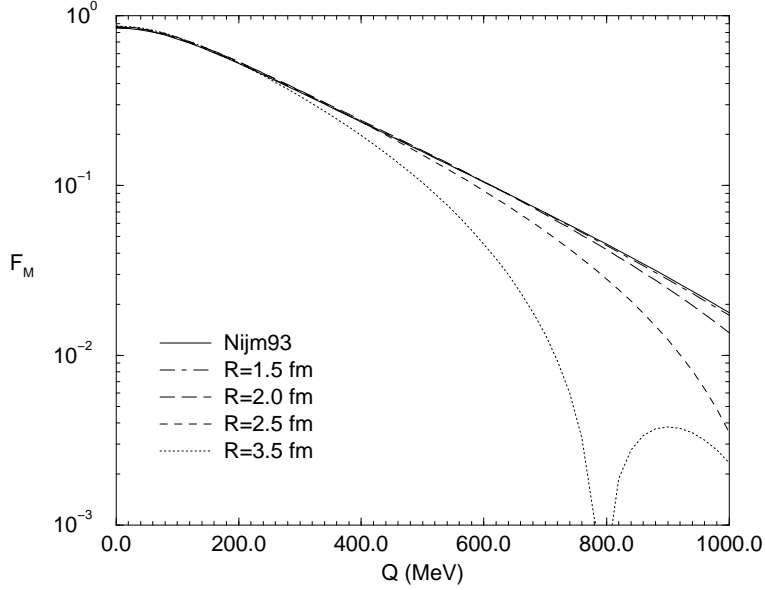


Figure 8: The magnetic form factor of the deuteron for several different wave functions, all of which have the same tail, and also the same one-pion exchange part. Legend as in Fig. 6.

The situation is somewhat different for the quadrupole form factor F_Q . As displayed in Fig. 10 there is a problem because the calculations with $R = 2.5$ fm and $R = 3.5$ fm do not reproduce the quadrupole moment of the Nijm93 calculation very well. On the other hand, the shape of all curves except the $R = 3.5$ fm one is basically correct out to about $Q = 500$ MeV. The reason for this discrepancy is clear if one examines the D-state wave functions for $R = 2.5$ fm. This wave function is a very poor approximation to the Nijm93 wave function for $R < 2.5$ fm, although it agrees with it almost exactly for $R \geq 2.5$ fm. Thus, we expect that the quadrupole form factor will be poorly described by such a wave function once Q is of the order of 300–400 MeV.

However, it should be noted that one can get a larger range of Q over which different calculations agree if one restricts $R \leq 2$ fm. In this case the low-momentum experimental data on the electron-deuteron tensor-polarization observable T_{20} is well reproduced by wave functions with a range of different R 's. From Fig. 11 it would seem that our lowest-order EFT calculation can, provided R is kept in an admittedly fairly small range, describe the data of Refs. [64]–[69] fairly well out to about $Q \approx 700$ MeV.

Note that we have not compared our results for A and B with experimental data. A brief look at our calculation suffices to show that the scale of breakdown in predicting A and B will have little to do with the nuclear physics of the problem. Instead, it will be set by the assumption of pointlike nucleons. Thus, calculations for A and B already do poorly with respect to the experimental data at $Q \approx 400$ MeV. This failure is not due to sensitivity to short-distance nuclear dynamics. Rather, it is due to the dominance of nucleon-structure corrections in the higher-order contributions to the Λ -counting kernel for this process. In contrast, in T_{20} the form factors of the nucleons essentially

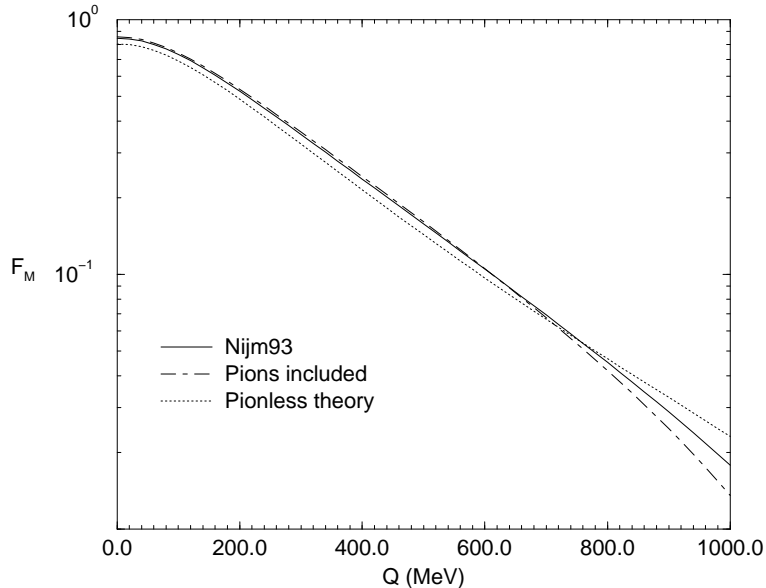


Figure 9: The magnetic form factor of the deuteron for a short-distance potential of $R = 2.0$ fm, both with and without one-pion exchange included. Legend as in Fig. 7.

cancel out, and so it *is* worthwhile to confront our NLO predictions with experimental data there ⁸.

Finally, Fig. 12 shows results in the pionless theory for F_Q . Since no tensor mixing is included in the pionless calculation the only way that the D-state enters is through the asymptotic D-to-S ratio. This physics is apparently not enough to reproduce Q_d , although the low-momentum shape of F_Q is essentially correct. Similar results to this have been seen in NNLO calculations of F_Q using the version of Q-counting which applies in the effective field theory without pions [12].

6 Summary and discussion

In this paper we have performed an effective field theory analysis of electron-deuteron scattering. In such an approach the Lagrangian of heavy-baryon chiral perturbation theory, which contains nucleons, pions, and photons as explicit fields, is employed to derive a diagrammatic expansion for any process. The naive engineering dimension of these diagrams can then be computed and this allows a particle-number-conserving Hamiltonian for the NN system to be written down as an expansion in operators of increasing dimensionality. Then, if an operator has dimensionality D relative to the leading contribution to some observable its physical effects are suppressed by $(P/\Lambda)^D$, where P is either the typical momentum in the problem, or m_π , and Λ is either the mass of the nucleon, or Λ_χ , the chiral symmetry breaking scale. This leads to an expansion for the NN potential in which the leading interaction is the sum of a one-pion exchange potential and a contact interaction [14, 15, 27, 24]. Corrections to this are suppressed by P^2/Λ^2 .

⁸We thank Rocco Schiavilla for pointing this out to us, and for suggesting we calculate T_{20} in this approach.

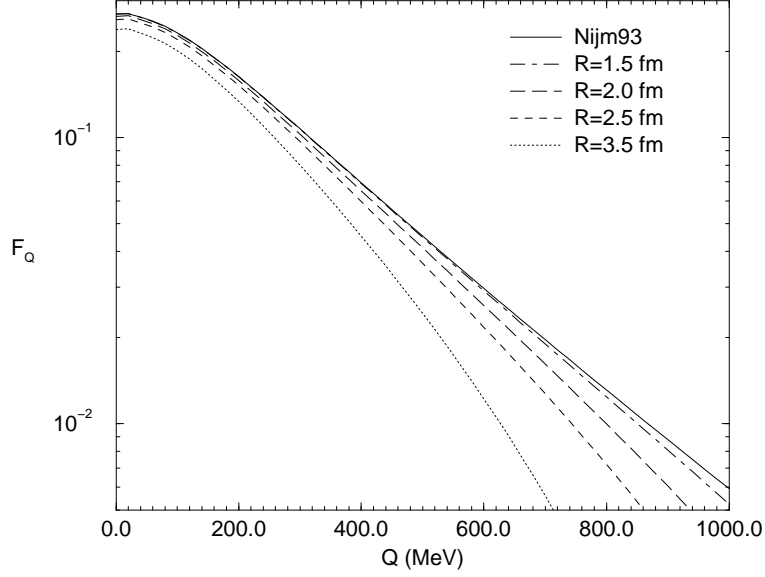


Figure 10: The quadrupole form factor of the deuteron for several different wave functions, all of which have the same tail, and also the same one-pion exchange part. Legend as in Fig. 6.

In this work we implemented the ideas of this “ Λ -counting” by demanding that the asymptotic deuteron wave function correspond to the correct deuteron binding energy and the experimental values of A_S and A_D (the asymptotic S- and D-state normalizations). This wave function was then integrated in from $r = \infty$ to finite r , using a Schrödinger equation which included the one-pion exchange potential. At an arbitrary radius R we then imposed a short-distance regulator, whose role is to give the deuteron wave function the appropriate $r \rightarrow 0$ limit. For $r > R$ corrections to the wave function thus obtained occur from higher-order effects in the Λ -counting expansion of the NN potential, and so are suppressed by P^2/Λ^2 . Effective field theory then leads us to hope that physical observables will not be sensitive to the artificial short-distance potential we have imposed. Indeed, the statement that observables should not depend on R is really the renormalization group for this formulation of effective field theory, with R playing the role of an inverse cutoff. The degree to which we have renormalization-group invariance, i.e. independence of physical quantities on R , can be checked *a posteriori*, as was done here. Our calculations showed that the charge and magnetic form factors of the deuteron are largely insensitive to the value of R for $Q \leq 700$ MeV. The quadrupole form factor is more sensitive to the short-distance physics, but is still essentially correct for $Q \leq 300$ MeV. Indeed, the shape of F_Q is independent of R for $Q \leq 700$ MeV, but the deuteron’s quadrupole moment is somewhat sensitive to the short-distance potential. The inclusion of a two-body counterterm for Q_d in the expansion of the electron-deuteron scattering kernel will remove this undue sensitivity to short-distance physics.

A consistent, systematic expansion for the electromagnetic current inside the deuteron is also a result of Λ -counting. Again, more complicated effects, including two-body currents, the effects of nucleon size, and relativistic corrections are suppressed by powers of P/Λ . Indeed, we showed that

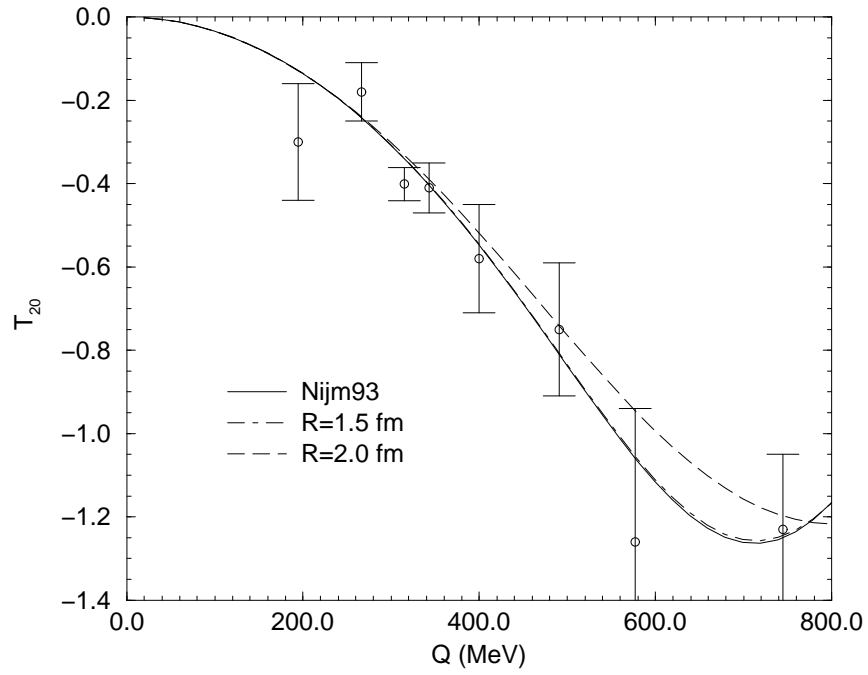


Figure 11: The tensor polarization observable T_{20} . The solid line is the result using the Nijm93 wave function, while the dot-dashed and long-dashed lines are calculations with $R=1.5$ fm and $R=2$ fm respectively.

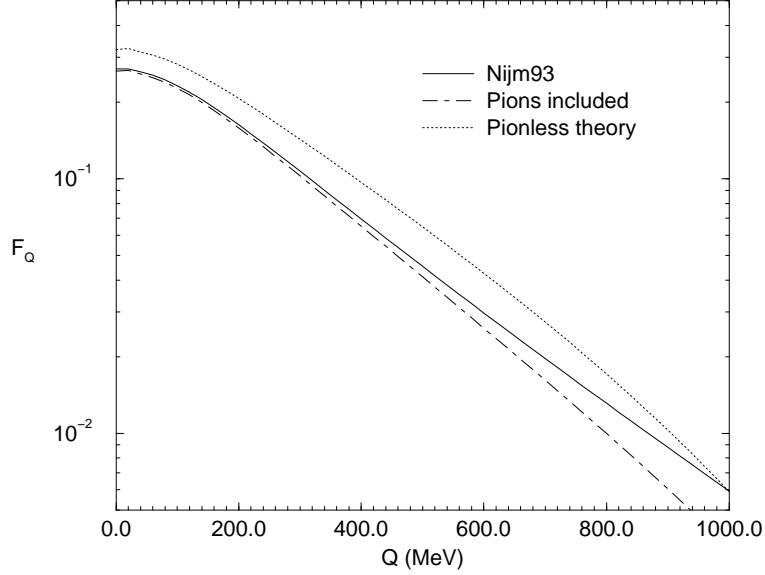


Figure 12: The quadrupole form factor of the deuteron for a short-distance potential of $R = 2.0$ fm, both with and without one-pion exchange included. Legend as in Fig. 7.

the impulse approximation with nucleons of charge e and zero, and magnetic moments μ_p and μ_n is the next-to-leading-order deuteron current in Λ -counting. Corrections are suppressed by powers of P^2/Λ^2 . We also found that the dominant corrections to this picture arise from the effects of the finite size of the nucleon, and from two-body currents. The leading two-body currents in this expansion have already been found to have a significant effect in potential-model calculations of electron-deuteron scattering. Thus it seems that Λ -counting can be used to systematize some of the findings of these calculations.

The approach adopted in this work allows for an accurate calculation of the long-distance quantities $\langle r^{2n} \rangle$. These are dominated by the tail of the wave function, which is precisely reproduced in our approach. Indeed, the deuteron is a shallow bound state, and so *most* of the wave function is in the tail. This observation suffices to show that the leading relative error in $\langle r^{2n} \rangle$ varies as $(\gamma R)^{2n+1}$. By contrast, for the electromagnetic deuteron form factors F_C , F_M and F_Q , the error estimate is only that our computation is accurate to $O(P^2/\Lambda^2)$. This leads us to observe that among the corrections arising at orders beyond that considered here, some are more important than others. For instance, relativistic corrections are suppressed by powers of γ/M and $2/(MR)$, and so turn out to be less important than corrections due to finite-nucleon size—at least once Q^2 is moderately large. This is a generic issue in effective field theory: the error estimate is always conservative. If the relative error is $(P/\Lambda)^2$ then the estimate is made by taking the largest possible scale for P and the smallest possible scale for Λ . One example of this is electron-deuteron scattering at low Q . Blind application of Λ -counting suggests an error of size $(m_\pi/\Lambda_\chi)^2$, but in fact the error in F_C for $Q \lesssim 2\gamma$ is much smaller than this, varying like $Q^2\gamma R^3$. Thus a sacrifice is made: efforts to make the power-counting simple may lead to a significant overestimate of the error of the calculation.

Of course, Λ -counting is not the only approach which has been proposed to EFT in nuclear physics. In Ref. [26] the electromagnetic deuteron form factors were calculated using the Q-counting scheme developed in Refs. [2, 6]. As far as direct error estimation goes Q-counting enjoys a significant practical advantage over Λ -counting. In Q-counting all physical amplitudes are expanded in Q/Λ with

$$Q \sim \gamma, m_\pi, p; \quad (76)$$

where p is either a nucleon momentum or the momentum of an external probe. Since Q-counting is an expansion for amplitudes themselves one has immediate formal error estimates. A calculation of a quantity to order Q^n beyond leading order implies an error of relative order $(Q/\Lambda)^{n+1}$. The EFT error estimate then is given by choosing the largest of γ , p , and m_π as Q and the smallest possible short-distance scale for Λ .

Within their common domain of applicability, employing Λ -counting to derive an NN potential V which is valid to order $(P/\Lambda)^n$ and then using the Lippmann-Schwinger equation for the NN amplitude to get T from V is no worse than expanding T to order $(Q/\Lambda)^n$. Looked at this way it becomes clear that Λ -counting includes additional contributions beyond the order to which one is working in Q-counting. Empirically, we see in both this and other work that calculations based on Λ -counting describe the data no worse than Q-counting calculations with the same number of parameters. Indeed, they often do better. For example, the deuteron static properties calculated here and in Ref. [28] are considerably more accurate than those of Ref. [26], yet all three calculations are to next-to-leading order. Similarly, here the deuteron form factors are well explained to considerably higher Q than in Ref. [26]. This suggests that the deuteron wave function is more accurately described at $r \sim 1/m_\pi$ if Λ -counting is employed. The physics which drives this improvement in the description of the form factors at $Q \sim 2m_\pi$ is apparently Λ -counting's use of a OPEP which is iterated to all orders.

In an attempt to elucidate this issue we have integrated the S-wave deuteron wave function in from $r = \infty$, including the effects of the central part of one-pion exchange only to leading order in perturbation theory. The results thereby obtained are quite different from the nonperturbative treatment of one-pion exchange employed in Section 3 (see Fig. 13). Indeed the resulting $u(r)$ differs significantly from the “true” wave function at $r \sim 2.5$ fm—well below the scale where the wave function with OPEP iterated to all orders starts to deviate markedly from $u(r)$ for the Nijm93 potential. The dominant effect which brings the short-dashed curve of Fig. 13 down to the long-dashed one is the tensor coupling of the D-wave piece of the deuteron back into the S-wave. This may well be an indication that iterating parts of the low-order EFT NN potential (such as one-pion exchange) to all orders generates effects which are numerically important even though they are formally of higher-order in Q-counting.

This possibility is not unreasonable since the dimensionless parameter identified with iterations of the one-pion exchange interaction in S-wave scattering is [2]:

$$\frac{m_\pi}{\Lambda_{NN}} = \frac{g_A^2 M m_\pi}{16\pi f_\pi^2} \approx 0.5, \quad (77)$$

a fairly large expansion parameter [30]. This is of particular concern in the triplet channel, since there are large Clebsch-Gordon coefficients in the S-D coupling OPEP. In addition, the form of the tensor force gives an enhancement factor relative to the central force of 7 at $r = 1/m_\pi$. These large enhancements underlie the conventional nuclear folklore that the second-order tensor force is extremely important, indeed far more important than central OPE. In contrast, in Q-counting, the effects of the tensor force on, say the 3S_1 phase shift only appear at next-to-next-to-leading

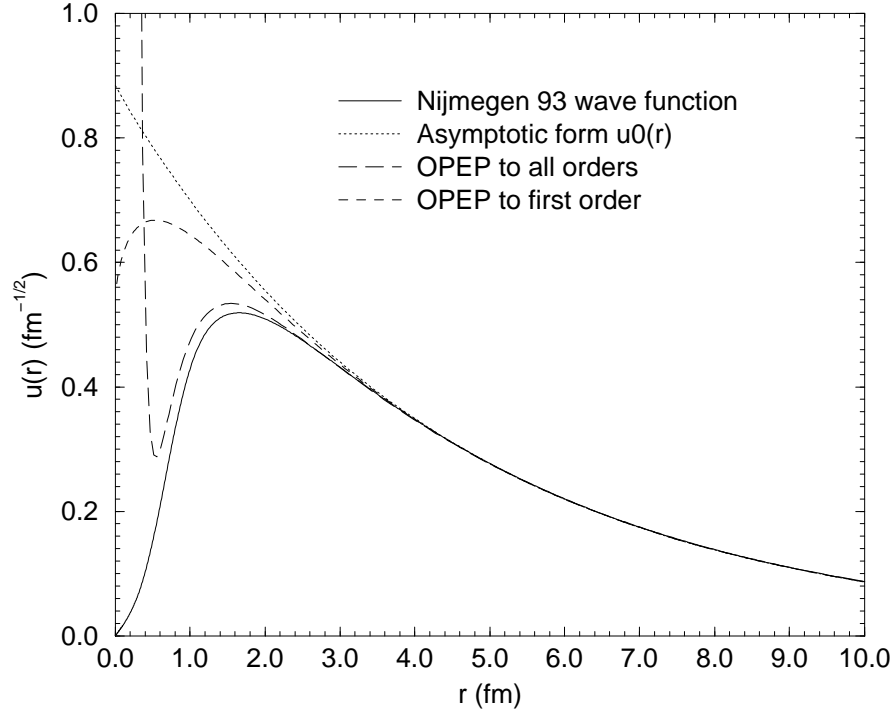


Figure 13: The S-wave radial wave function of the deuteron, $u(r)$. The Nijm93 wave function is the solid line, the asymptotic form integrated in to $r = 0$ without one-pion exchange gives the dotted line, while including one-pion exchange when integrating in yields the long-dashed line. The short-dashed line is the result when the central part of one-pion exchange is included only to first order in perturbation theory.

order. This raises questions about how efficiently Q-counting accounts for these effects. In contrast, Λ -counting includes this physics in a low-order calculation. While none of this demonstrates that Q-counting fails it sheds some doubt on the desirability of treating one-pion exchange effects perturbatively. A calculation of NN phase shifts in the ${}^3S_1 - {}^3D_1$ channel to NNLO in Q-counting is near completion [70]. Clearly, such further investigation of the efficacy of a perturbative expansion for pionic physics is very important.

What is also clear is that Λ -counting leads to an effective field theory approach for electromagnetic deuteron properties which is viable for momenta Q up to at least 700 MeV. The picture that emerges is not very different from that seen in NN potential model calculations of the same quantities. The main advantage of an effective field theory treatment at these momentum transfers is that corrections to both the NN interaction and the deuteron current appear as systematic corrections in an expansion in

$$\frac{(p, m_\pi, Q)}{(\Lambda_\chi, M)}, \quad (78)$$

where p is the typical momentum inside the deuteron and Λ_χ is the scale of chiral symmetry breaking.

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